GCE A Level Maths 9709

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1.4 Circular Measure

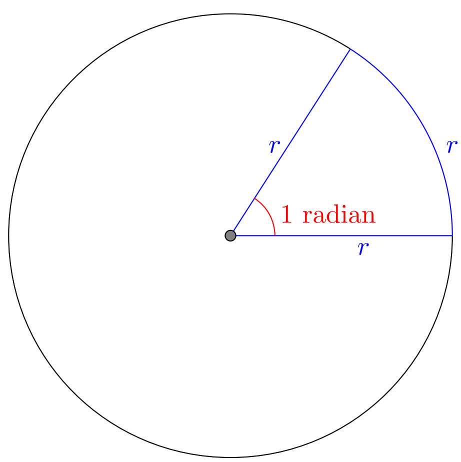
In this topic we will learn to:

- understand the definition of a radian, and use the relationship between radians and degrees
- use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ in solving problems concerning the arc length and sector area of a circle

Circular Measure

Radian

A radian is an angle whose corresponding arc in a circle is equal to the radius of the circle.



One radian is ≈ 57.2958 **degrees.**

To change an angle from radians to degrees, we use the formula,

$$\theta = \text{angle in radians} \times \frac{180^\circ}{\pi}$$

To change an angle from degrees to radians, we use the formula,

$$\theta = \text{angle in degrees} \times \frac{\pi}{180^\circ}$$

Arc Length

An arc is a portion of the circumference of the circle. The length of that portion is called the arc length. Arc length is represented by the symbol, s. To calculate the arc length, we use the formula,

$$s = r\theta$$

Where r is the radius of the circle and θ is the angle of the sector.

Sector Area of a Circle

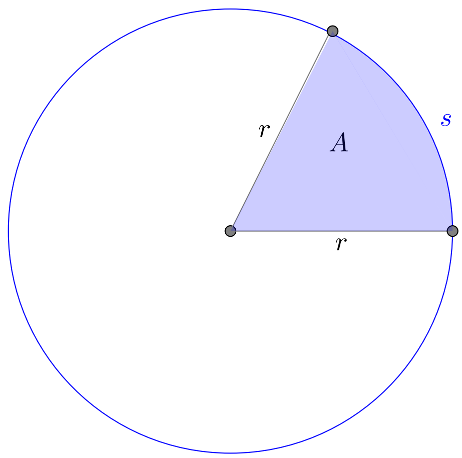
A sector of a circle is a pie shaped portion of a circle, consisting of an arc and its two radii. To calculate the sector area of circle, we use the formula,

$$A = \frac{1}{2}r^2\theta$$

Note: The formulae for arc length and sector area of a circle take θ in radians NOT in degrees.

Let's look at some past paper questions on this topic.

1. A sector of a circle of radius r cm has an area of A cm^2. Express the perimeter of the sector in terms of r and A. (9709/11/M/J/19 number 3)



Perimeter is the distance around the shape,

$$P = r + s + r$$

Using the formula for arc length, substitute s with $r\theta$,

$$P = r + r\theta + r$$

$$P = 2r + r\theta$$

Use the formula for sector area of a circle, to get rid of θ,

$$A = \frac{1}{2}r^2\theta$$

Make θ the subject of the formula,

$$\theta = \frac{2A}{r^2}$$

Substitute θ,

$$P = 2r + r\theta$$

$$P = 2r + r\left(\frac{2A}{r^2}\right)$$

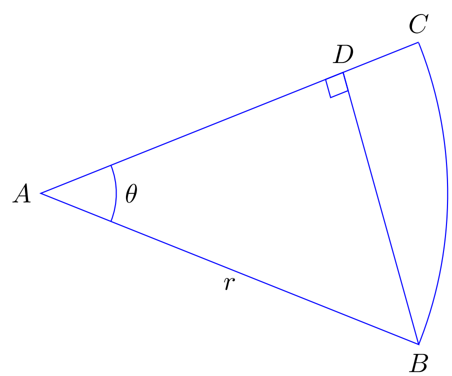
Simplify,

$$P = 2r + \frac{2A}{r}$$

Therefore, the final answer is,

$$P = 2r + \frac{2A}{r}$$

2. The diagram shows a sector ABC of a circle with center A and radius r. The line BD is perpendicular to AC. Angle CAB is θ radians. (9709/11/M/J/22 number 5)



(a) Given that $\theta = \frac{1}{6}\pi$, find the exact area of BCD in terms of r.

If you look at the diagram, you will notice that the Area of BCD can be written as,

$$\text{Area of } BCD = \text{Area of Sector } ABC - \text{ Area of triangle } ABD$$

Use the formula for sector area of a circle and area of a triangle,

$$\text{Area of } BCD = \frac{1}{2}r^2\theta - \frac{1}{2} \times BD \times AD$$

Substitute in the value of θ,

$$\text{Area of } BCD = \frac{1}{2}r^2\left(\frac{1}{6}\pi\right) - \frac{1}{2} \times BD \times AD$$

Simplify,

$$\text{Area of } BCD = \frac{1}{12}\pi r^2\theta - \frac{1}{2} \times BD \times AD$$

Let's evaluate BD using Pythagoras,

$$\sin\theta = \frac{BD}{r}$$

$$BD = r\sin\theta$$

Since $\theta = \frac{1}{6}$, then,

$$BD = r\sin\left(\frac{1}{6}\pi\right)$$

$$BD = \frac{1}{2}r$$

Let's evaluate AD using Pythagoras,

$$\cos\theta = \frac{AD}{r}$$

$$AD = r\cos\theta$$

$$AD = r\cos\left(\frac{1}{6}\pi\right)$$

$$AD = \frac{\sqrt{3}}{2}r$$

Let's substitute BD and AD,

$$\text{Area of } BCD = \frac{1}{12}\pi r^2\theta - \frac{1}{2} \times BD \times AD$$

$$\text{Area of } BCD = \frac{1}{12}\pi r^2\theta - \frac{1}{2} \times \frac{1}{2}r \times \frac{\sqrt{3}}{2}r$$

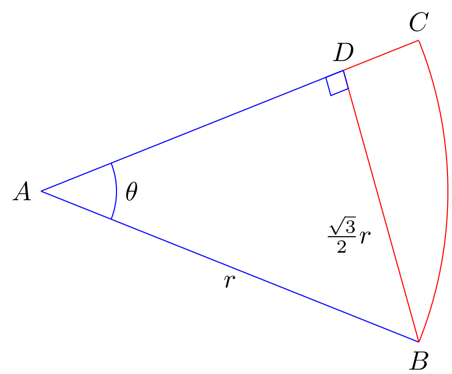
Simplify,

$$\text{Area of } BCD = \frac{1}{12}\pi r^2\theta - \frac{\sqrt{3}}{8}r^2$$

Therefore, the final answer is,

$$\text{Area of } BCD = \frac{1}{12}\pi r^2\theta - \frac{\sqrt{3}}{8}r^2$$

(b) Given instead that the length of BD is $\frac{\sqrt{3}}{2}r$, find the exact perimeter of BCD in terms of r.



Perimeter is the distance around the shape,

$$P = BD + CD + arcBC$$

Substitute BD with $\frac{\sqrt{3}}{2}r$,

$$P = \frac{\sqrt{3}}{2}r + CD + arcBC$$

From the diagram, we can tell that $CD = AC - AD$,

$$P = \frac{\sqrt{3}}{2}r + (AC - AD) + arcBC$$

From part (a) we know that $AD = r\cos\theta$ and AC is the radius,

$$P = \frac{\sqrt{3}}{2}r + (r - r\cos\theta) + arcBC$$

Using the formula for arc length, substitute $arcBC$ with $r\theta$,

$$P = \frac{\sqrt{3}}{2}r + (r - r\cos\theta) + r\theta$$

We can use Pythagoras to evaluate θ,

$$\sin\theta = \frac{BD}{AB}$$

Note: The right-angled triangle allows us to use SOHCA-TOA.

$$\sin\theta = \frac{\frac{\sqrt{3}}{2}r}{r}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\theta = \frac{1}{3}\pi$$

Substitute θ,

$$P = \frac{\sqrt{3}}{2}r + (r - r\cos\theta) + r\theta$$

$$P = \frac{\sqrt{3}}{2}r + \left(r - r\cos\left(\frac{1}{3}\pi\right)\right) + r\left(\frac{1}{3}\pi\right)$$

Simplify,

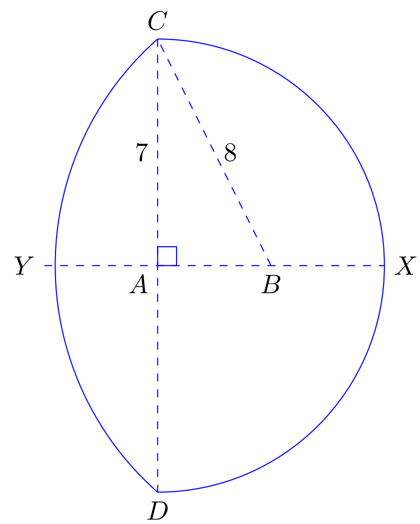
$$P = \frac{\sqrt{3}}{2}r + \left(r - \frac{1}{2}r\right) + \frac{1}{3}\pi r$$

$$P = \frac{\sqrt{3}}{2}r + \frac{1}{2}r + \frac{1}{3}\pi r$$

Therefore, the final answer is,

$$P = \frac{\sqrt{3}}{2}r + \frac{1}{2}r + \frac{1}{3}\pi r$$

3. In the diagram, CXD is a semicircle of radius 7 cm with center A and diameter CD. The straight line $YABX$ is perpendicular to CD, and the arc CYD is part of a circle with center B and radius 8 cm. Find the total area of the region enclosed by the two arcs. (9709/12/F/M/19 number 3)



From the diagram, we can tell that the total area can be written as,

$$\text{Total Area } = \text{Area of semicircle } CXD + \text{Area of segment } CYD$$

Let's start by finding the Area of semicircle CXD,

$$\text{Area of semicircle } CXD = \frac{1}{2}\pi r^2$$

CXD has a radius of 7 cm,

$$\text{Area of semicircle } CXD = \frac{1}{2}\pi(7)^2$$

Simplify,

$$\text{Area of semicircle } CXD = \frac{49}{2}\pi$$

Let's find the Area of segment CYD,

$$\text{Area of segment } CYD = \text{Area of sector } CYD - \text{Area of triangle } BCD$$

Use formulae for sector area of a circle and area of a triangle,

$$\text{Area of segment } CYD = \frac{1}{2}r^2\theta - \frac{1}{2} \times CD \times AB$$

Substitute in r and CD,

$$\text{Area of segment } CYD = \frac{1}{2} \times 8^2 \times \theta - \frac{1}{2} \times 14 \times AB$$

Simplify,

$$\text{Area of segment } CYD = 32\theta - 7 \times AB$$

Let's find AB using Pythagoras,

$$a^2 + b^2 = c^2$$

$$(AB)^2 + (AC)^2 = (BC)^2$$
$$(AB)^2 + 7^2 = 8^2$$
$$(AB)^2 = 8^2 - 7^2$$
$$(AB)^2 = 15$$
$$AB = \sqrt{15}$$

Substitute AB and simplify,

$$\text{Area of segment } CYD = 32\theta - 7 \times AB$$

$$\text{Area of segment } CYD = 32\theta - 7 \times \sqrt{15}$$
$$\text{Area of segment } CYD = 32\theta - 7\sqrt{15}$$

Let's find θ,

$$\theta = 2ABC$$

$$\sin(ABC) = \frac{AC}{BC}$$
$$\sin(ABC) = \frac{7}{8}$$
$$ABC = \sin^{-1}\left(\frac{7}{8}\right)$$

$$\theta = 2\sin^{-1}\left(\frac{7}{8}\right)$$

Substitute θ and simplify,

$$\text{Area of segment } CYD = 32\theta - 7\sqrt{15}$$

$$\text{Area of segment } CYD = 32\left(2\sin^{-1}\left(\frac{7}{8}\right)\right) - 7\sqrt{15}$$

$$\text{Area of segment } CYD = 64\sin^{-1}\left(\frac{7}{8}\right) - 7\sqrt{15}$$

Therefore, the total area is,

$$\text{Total Area } = \text{Area of semicircle } CXD + \text{Area of segment } CYD$$

$$\text{Total Area } = \frac{49}{2}\pi + 64\sin^{-1}\left(\frac{7}{8}\right) - 7\sqrt{15}$$

$$\text{Total Area } = 118.0$$

Note: Remember that θ is in radians, so evaluate any trig functions in radians.

Therefore, the final answer is,

$$\text{Total Area } = 118.0$$