GCE A Level Maths 9709

SMIYL

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1.7 Differentiation

In this topic we will learn how to:

- use the notations
$$f'(x), \quad f''(x), \quad \frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$
- use the derivative of x^n (for any rational n), together with constant multiples, sums and differences of functions, and of composite functions using the chain rule

Basic Differentiation

Differentiation is a form of calculus in which we determine the derivative of any function. The derivative of any function is its gradient, it is denoted by,

$$\frac{dy}{dx}$$

If the function is denoted by $f(x)$ then $\frac{dy}{dx}$ is written as,

$$f'(x)$$

Once you find the first derivative of a function, you have also found the gradient function.

The Second Derivative

If you differentiate the first derivative, you get the second derivative. This is denoted by,

$$\frac{d^2y}{dx^2}$$

For the function $f(x)$, it would be written as,

$$f''(x)$$

The second derivative is used to determine the nature of stationary points i.e whether a stationary point is a maximum or minimum turning point.

Differentiation of ax^n

To differentiate $y = ax^n$, where a is a non-zero constant we use the general formula,

$$\frac{dy}{dx} = anx^{n-1}$$

Note: If you differentiate a constant, you get zero.

Differentiation of $(ax+b)^n$

To differentiate $y = (ax+b)^n$ we use the general formula,

$$\frac{dy}{dx} = an(ax+b)^{n-1}$$

This is the formula we get from the chain rule, however, for more complex functions it may be wiser to use the chain rule.

The Chain Rule

The chain rule gives us a way to calculate the derivative of a composition of functions. For example, if we want to find the derivative of $y = (2x+1)^3$,

$$y = (2x+1)^3$$

First identify the inner function i.e whatever is inside the bracket, and equate it to u,

$$u = 2x+1$$

Substitute u into the original equation,

$$y = (2x+1)^3$$

$$y = u^3$$

This is the outer function.

We now have two separate equations, which we can differentiate easily,

$$y = u^3 \qquad u = 2x+1$$

Before we differentiate, we can define a chain rule. A chain rule, is an equation that allows to find our required differential, by considering the derivatives of the inner and outer functions separately. To define a chain rule, start by writing out the derivative you want to find,

$$\frac{dy}{dx}$$

Equate it to the product of the derivatives of the inner and outer functions,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Note: If you were to simplify the right-hand side, you would get $\frac{dy}{dx}$.

Now let's differentiate the two equations to evaluate $\frac{dy}{du}$ and $\frac{du}{dx}$,

$$y = u^3 \qquad u = 2x + 1$$

$$\frac{dy}{du} = 3u^2 \qquad \frac{du}{dx} = 2$$

Now we can use the chain rule, we defined earlier,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitute into the chain rule,

$$\frac{dy}{dx} = 3u^2 \times 2$$

Which simplifies to give,

$$\frac{dy}{dx} = 6u^2$$

Replace u,

$$\frac{dy}{dx} = 6(2x + 1)^2$$

Your final differential would be,

$$\frac{dy}{dx} = 6(2x + 1)^2$$

Let's walk through some examples to cement the concepts we have learnt above.

1. Differentiate $y = 5x^4 + 2x^3 + 3x^2 + x + 7$.

$$y = 5x^4 + 2x^3 + 3x^2 + x + 7$$

Let's start by differentiating the first term,

$$4(5)x^{4-1}$$

$$20x^3$$

Let's move on to the second term,

$$y = 5x^4 + 2x^3 + 3x^2 + x + 7$$

Let's differentiate the second term,

$$3(2)x^{3-1}$$

$$6x^2$$

Let's move on to the third term,

$$y = 5x^4 + 2x^3 + 3x^2 + x + 7$$

Let's differentiate the third term,

$$2(3)x^{2-1}$$

$$6x$$

Let's move on to the fourth term,

$$y = 5x^4 + 2x^3 + 3x^2 + x + 7$$

Let's differentiate the fourth term,

$$(1)x^{1-1}$$

$$1$$

The fifth term, 7, gives a 0 when it is differentiated. Therefore, putting everything together, the final answer would be,

$$\frac{dy}{dx} = 20x^3 + 6x^2 + 6x + 1$$

2. It is given that the function f is defined by $f(x) = \frac{4}{x^{\frac{1}{2}}} - \frac{2}{3x^{\frac{3}{2}}}$. Find $f'(x)$.

$$f(x) = \frac{4}{x^{\frac{1}{2}}} - \frac{2}{3x^{\frac{3}{2}}}$$

The first step, is to use the laws of indices to bring up the parts containing x,

$$f(x) = 4x^{-\frac{1}{2}} - \frac{2}{3}x^{-\frac{3}{2}}$$

Note: We cannot differentiate if any part containing x is in the denominator.

Now we can differentiate,

$$f(x) = 4x^{-\frac{1}{2}} - \frac{2}{3}x^{-\frac{3}{2}}$$

Let's start with the first term,

$$-\frac{1}{2}(4)x^{-\frac{1}{2}-1}$$

$$-2x^{-\frac{3}{2}}$$

Let's move on to the second term,

$$f(x) = 4x^{-\frac{1}{2}} - \frac{2}{3}x^{-\frac{3}{2}}$$

Note: The negative sign is part of the second term.

Let's differentiate the second term,

$$-\left(-\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{-\frac{3}{2}-1}$$

$$x^{-\frac{5}{2}}$$

Putting everything together,

$$f'(x) = -2x^{-\frac{3}{2}} + 3x^{-\frac{5}{2}}$$

Note: It is adviseable to make the powers in x positive, however, it is not mandatory.

Therefore, the final answer would be,

$$f'(x) = -\frac{2}{x^{\frac{3}{2}}} + \frac{1}{x^{\frac{5}{2}}}$$

3. It is given that $y = \sqrt{2x^3 + 5}$. Find $\frac{dy}{dx}$.

$$y = \sqrt{2x^3 + 5}$$

Let's use laws of indices to rewrite the square root sign as the power $\frac{1}{2}$,

$$y = (2x^3 + 5)^{\frac{1}{2}}$$

To differentiate this function we need to use the chain rule. Let's identify the inner function and equate it to u,

$$u = 2x^3 + 5$$

Let's write the outer function in terms of u,

$$y = (2x^3 + 5)^{\frac{1}{2}}$$

$$y = u^{\frac{1}{2}}$$

We now have two functions that are easy to differentiate,

$$y = u^{\frac{1}{2}} \qquad u = 2x^3 + 5$$

Let's use the derivatives of the two functions to define a chain rule for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Let's differentiate our two functions to evaluate $\frac{dy}{du}$ and $\frac{du}{dx}$,

$$y = u^{\frac{1}{2}} \qquad u = 2x^3 + 5$$

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} \qquad \frac{du}{dx} = 3(2)x^{3-1}$$

$$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} \qquad \frac{du}{dx} = 6x^2$$

Let's go back to the chain rule we defined earlier,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Substitute in the expressions of $\frac{dy}{du}$ and $\frac{du}{dx}$,

$$\frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times 6x^2$$

Simplify,

$$\frac{dy}{dx} = 3x^2u^{-\frac{1}{2}}$$

Substitute u,

$$\frac{dy}{dx} = 3x^2(2x^3 + 5)^{-\frac{1}{2}}$$

Therefore, the final answer is,

$$\frac{dy}{dx} = 3x^2(2x^3 + 5)^{-\frac{1}{2}}$$