GCE A Level Maths 9709

SMIYL

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1.7 Differentiation

In this topic we will learn how to:

- apply differentiation to tangents and normals

Tangents and Normals

When we differentiate a curve, we are finding the gradient function of that curve. If we know the coordinates of that point, we can substitute them into the gradient function, to get the gradient. That gradient represents the gradient of a tangent through that particular point. Therefore, $\frac{dy}{dx}$ represents the gradient of the tangent. A normal is a line that is perpendicular to the tangent. To find the gradient of a normal, we have to first find the gradient of the tangent, then use the idea that,

$$m_1 \times m_2 = -1$$

Let's look at some past paper questions involving tangents and normals.

1. A curve has the equation $y = \frac{1}{(3x-2)^{\frac{3}{2}}}$. The normal to the curve at the point $(1, 1)$ crosses the y-axis at the point A. Find the y-coordinate of A. (9709/11/O/N/21 number 10)

$$y = \frac{1}{(3x-2)^{\frac{3}{2}}}$$

Take the denominator to the top using laws of indices,

$$y = (3x-2)^{-\frac{3}{2}}$$

Differentiate the equation of the curve,

$$\frac{dy}{dx} = \frac{3}{2} \times 3 \times (3x-2)^{-\frac{5}{2}}$$
$$\frac{dy}{dx} = \frac{9}{2}(3x-2)^{-\frac{5}{2}}$$

Substitute the x-coordinate at $(1,1)$ into the gradient function,

$$\frac{9}{2}(3(1)-2)^{-\frac{5}{2}} = -\frac{9}{2}$$

Therefore, the gradient of the tangent at $(1,1)$ is,

$$-\frac{9}{2}$$

Therefore, the gradient of the normal is,

$$\frac{2}{9}$$

Note: $m_1 \times m_2 = -1$ means the gradient of the tangent is the negative reciprocal of the gradient of the normal i.e to get the gradient of the normal, flip the gradient of the tangent and change its sign. If this is too complex, simply use $m_1 \times m_2 = -1$.

Now let's find the equation of the normal,

$$\textbf{passing through } (1,1) \qquad \textbf{gradient is } \frac{2}{9}$$

$$y = mx + c$$
$$1 = \left(\frac{2}{9}\right)(1) + c$$
$$1 = \frac{2}{9} + c$$
$$c = 1 - \frac{2}{9}$$
$$c = \frac{7}{9}$$

Therefore, the equation of the normal is,

$$y = \frac{2}{9}x + \frac{7}{9}$$

At A, $x = 0$,

$$y = \frac{2}{9}x + \frac{7}{9}$$
$$y = \frac{2}{9}(0) + \frac{7}{9}$$
$$y = \frac{7}{9}$$

Therefore, the y-coordinate at A is,

$$y = \frac{7}{9}$$

2. A curve has the equation $y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$. The curve has a stationary point $A(4, 0)$. Find the equation to the tangent at A. (9709/12/F/M/21 number 11)

$$y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$$

Differentiate the equation,

$$\frac{dy}{dx} = 9\left(-\frac{1}{2}x^{-\frac{3}{2}} + \frac{3}{2}(4)x^{-\frac{5}{2}}\right)$$
$$\frac{dy}{dx} = 9\left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x^{-\frac{5}{2}}\right)$$

Substitute the x-coordinate at $A(4, 0)$,

$$\frac{dy}{dx} = 9\left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x^{-\frac{5}{2}}\right)$$
$$\frac{dy}{dx} = 9\left(-\frac{1}{2}(4)^{-\frac{3}{2}} + 6(4)^{-\frac{5}{2}}\right)$$
$$\frac{dy}{dx} = \frac{9}{8}$$

Therefore, the gradient of the tangent is,

$$\frac{9}{8}$$

Let's find the equation of the tangent,

$$A(4,0) \qquad \textbf{gradient is } \frac{9}{8}$$

$$y = mx + c$$

$$0 = \frac{9}{8}(4) + c$$

$$0 = \frac{9}{2} + c$$

$$c = -\frac{9}{2}$$

Therefore, the equation of the tangent is,

$$y = \frac{9}{8}x - \frac{9}{2}$$