GCE A Level Maths 9709

SMIYL

April 2023

1.8 Integration

In this topic we will learn how to:

- use definite integration to find a volume of revolution about one of the axes

Volume of Revolution

We can use integration to find the volume produced by a graph when it rotates about an axis $360°$**. This is known as the volume of revolution. To find this we use the two formulae below,**

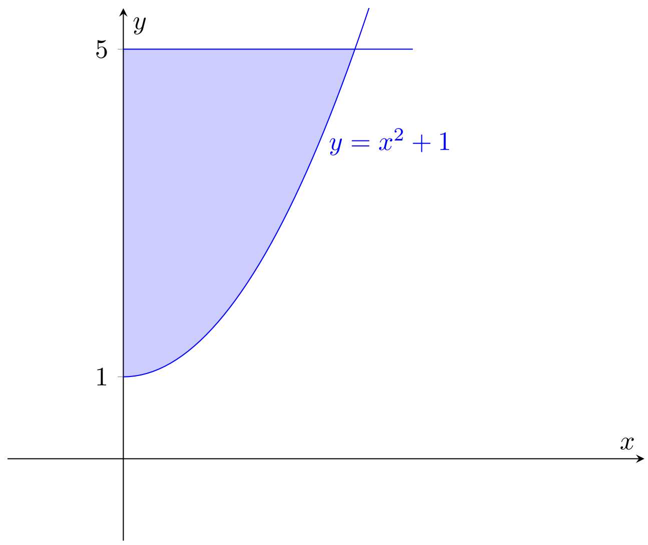
$$V = \int \pi y^2 \, dx$$

$$V = \int \pi x^2 \, dy$$

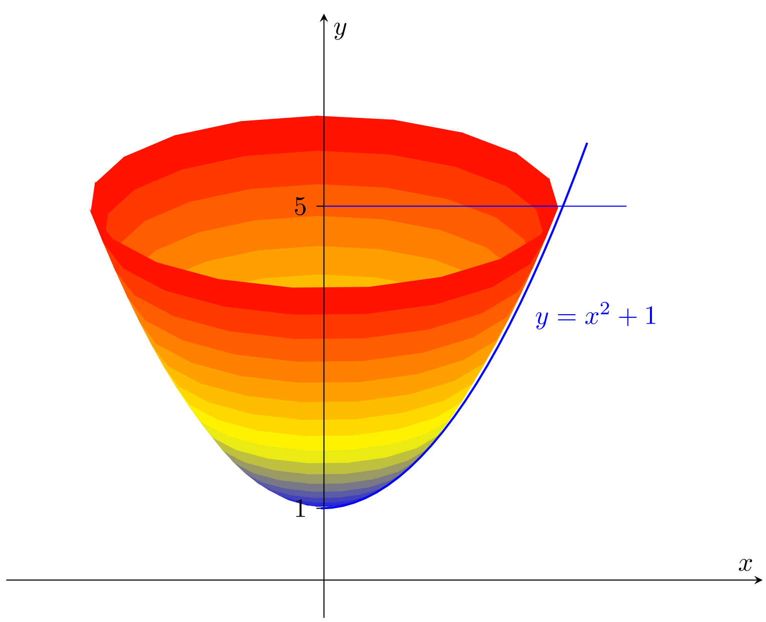
It is derived from the formula of volume of a cylinder, $V = \pi r^2 \, h$**, hence the similarities. The first equation is used when we're rotating about the** x**-axis. The second equation is used when we're rotating about the** y**-axis.**

Let's apply this to some past paper questions.

1. The diagram shows part of the curve with equation $y = x^2 + 1$. The shaded region enclosed by the curve, the y-axis and the line $y = 5$ is rotated through $360°$ about the y-axis. Find the volume obtained. (9709/12/F/M/20 number 3)



The volume of revolution of this graph would look like this,



$$y = x^2 + 1$$

Since we're rotating about the y-axis we will use the formula,

$$V = \int \pi x^2 \; dy$$

This means we need to find x^2 in terms of y since we are working with respect to y,

$$y = x^2 + 1$$

$$x^2 = y - 1$$

Let's substitute in x^2,

$$V = \int \pi x^2 \; dy$$

$$V = \int \pi(y - 1) \; dy$$

We are already given the limits in terms of y, so let's substitute them in,

$$V = \int_1^5 \pi(y - 1) \; dy$$

Note: If the limits are not in terms of y and you're rotating about the y-axis, use the equation of the curve, to convert them to be in terms of y.

Now let's integrate,

$$\int_1^5 \pi(y - 1) \; dy$$

pi **is a constant so you can move it outside the integral sign to make the integration easier,**

$$\pi \int_1^5 (y - 1) \; dy$$

$$\pi \left[\frac{y^2}{2} - y \right]_1^5$$

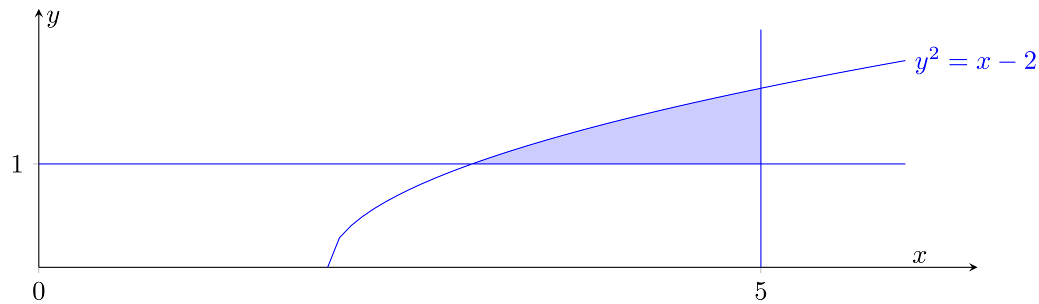
Substitute in the limits,

$$\pi\left[\left(\frac{(5)^2}{2}-5\right)-\left(\frac{(1)^2}{2}-1\right)\right]$$
$$\pi\left[\frac{15}{2}-\left(-\frac{1}{2}\right)\right]$$
$$\pi\left[\frac{15}{2}+\frac{1}{2}\right]$$
$$8\pi$$

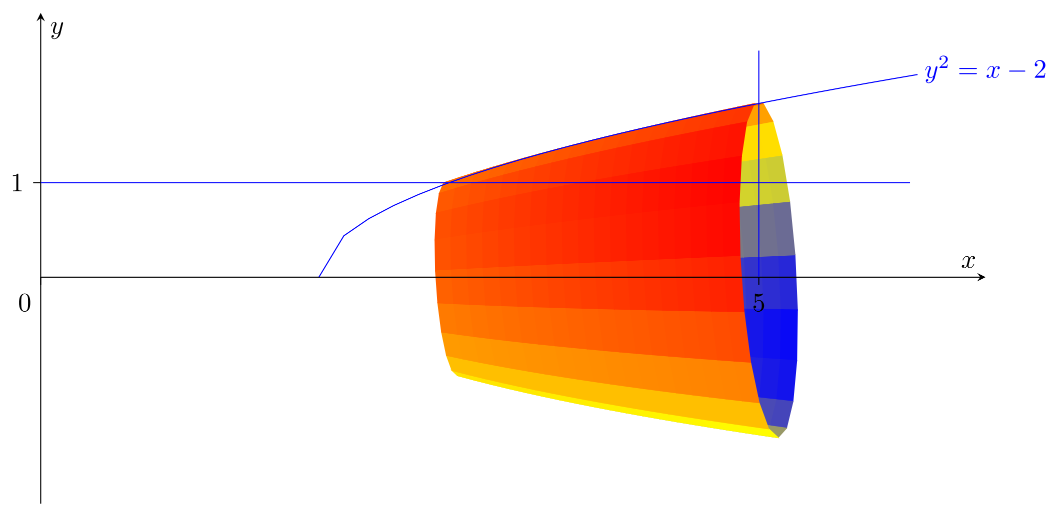
Therefore, the final answer is,

$$V=8\pi$$

2. The diagram shows part of the curve with equation $y^2 = x - 2$ and the lines $x = 5$ and $y = 1$. The shaded region is enclosed by the curve and the lines is rotated through 360° about the x-axis. Find the volume obtained. (9709/12/M/J/21 number 9)



The volume of revolution would look something like this,



Notice that we only want the shaded region which is bounded by the curve and the line $y = 1$. For that reason, we also need to consider the cylinder formed by the line $y = 1$ when it rotates,

$$V = \int \pi y^2 \; dx - \textbf{Area of cylinder}$$

Let's first find the volume produced by the curve,

$$\int \pi y^2 \; dx$$

Substitute in y^2,

$$\int \pi(x-2)\ dx$$

From the diagram, we can tell that one of our limits is 5. The other limit is at $y = 1$, so we need to convert it to be in terms of x,

$$\textbf{At } y = 1$$

$$y^2 = x - 2$$

$$1^2 = x - 2$$

$$1 = x - 2$$

$$x = 3$$

Therefore, our limits are 3 and 5,

$$\int_3^5 \pi(x-2)\ dx$$

Now let's integrate,

$$\pi \int_3^5 (x-2)\ dx$$

$$\pi \left[\frac{x^2}{2} - 2x\right]_3^5$$

Substitute in the limits,

$$\pi \left[\left(\frac{(5)^2}{2} - 2(5)\right) - \left(\frac{(3)^2}{2} - 2(3)\right)\right]$$

$$\pi \left[\frac{5}{2} - \left(-\frac{3}{2}\right)\right]$$

$$\pi \left[\frac{5}{2} + \frac{3}{2}\right]$$

$$4\pi$$

Now let's find the volume produced by the cylinder due to the line $y = 1$,

$$V = \pi r^2\ h$$

$$V = \pi \times 1^2 \times (5-3)$$

$$V = 2\pi$$

Therefore,

$$V = 4\pi - 2\pi$$
$$V = 2\pi$$

Therefore, the final answer is,

$$V = 2\pi$$