GCE A Level Maths 9709

SMIYL

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1.1 Quadratics

In this topic we will learn how to:

- find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant

The Discriminant

For the quadratic $ax^2 + bx + c$**,**

$$b^2 - 4ac$$

is known as the discriminant.

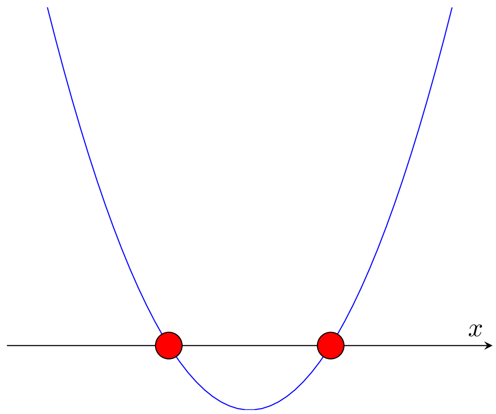
The discriminant is used to determine the number of roots that a quadratic has and the nature of those roots. A quadratic either has two real roots, one real root(repeated root), or no real roots.

Two real roots

If a quadratic has two real roots then,

$$b^2 - 4ac > 0$$

This means that the curve intersects the x-axis twice when sketched on a Cartesian plane.



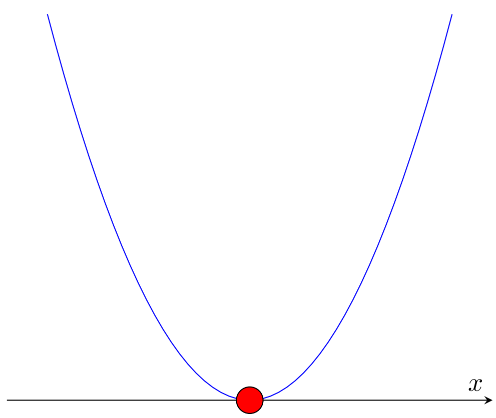
The above diagram, shows an example of a curve with two real roots

One Real Root (Repeated Root)

If a quadratic has one real root then,

$$b^2 - 4ac = 0$$

This means that the curve intersects the x-axis once when sketched on a Cartesian plane.



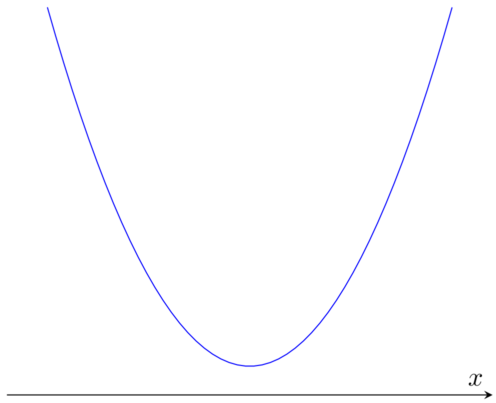
The above diagram, shows an example of a curve with one real root

No Real Roots

If a quadratic has no real roots then,

$$b^2 - 4ac < 0$$

This means that the curve does not intersect with the x-axis when sketched on a Cartesian plane.



The above diagram, shows an example of a curve with no real roots

Let's look at some past paper questions involving the discriminant.

1. The line $4y = x + c$, where c is a constant, is a tangent to the curve $y^2 = x + 3$ at the point P on the curve. Find the value of c.(9709/11/M/J/19 number 2)

   Gleaning information from the question we know that the line is tangent to the curve. This means the line intersects the curve once, therefore,

   $$b^2 - 4ac = 0$$

   Since the two graphs are intersecting, we start by solving simultaneously,

   $$4y = x + c \qquad y^2 = x + 3$$

   Make x the subject of the formula in the linear equation,

   $$4y = x + c$$

   $$x = c - 4y$$

Substitute x in the quadratic with $4y - c$,

$$y^2 = x + 3$$
$$y^2 = 4y - c + 3$$

Rearrange the equation so that all the terms are on one side,

$$y^2 - 4y + c - 3 = 0$$

Earlier, we deduced that,

$$b^2 - 4ac = 0$$
$$a = 1, \quad b = -4, \quad c = c - 3$$

Substitute these values into $b^2 - 4ac = 0$,

$$(-4)^2 - 4(1)(c - 3) = 0$$

Simplify,

$$16 - 4(c - 3) = 0$$

Remove the brackets,

$$16 - 4c + 12 = 0$$

Solve for c,

$$4c = 28$$
$$c = 7$$

Therefore, the final answer is,

$$c = 7$$

2. A line has equation $y = 3kx - 2k$ and a curve has equation $y = x^2 - kx + 2$, where k is a constant. Find the set of values of k for which the line and the curve meet at two distinct points.(9709/13/O/N/19 number 6)

   Let's glean some information from the question. It mentions "...set of values...". This means our answer should be a range of values, therefore, our answer is likely to be an inequality. The question also mentions that "...the line and the curve meet at two distinct points." This means they intersect twice and there are two real roots, therefore,

$$b^2 - 4ac > 0$$

So let's start by solving simultaneously since the two graphs intersect,

$$y = 3kx - 2k \qquad\qquad y = x^2 - kx + 2$$

Equate the two equations,

$$3kx - 2k = x^2 - kx + 2$$

Group like terms,

$$x^2 - kx - 3kx + 2 + 2k$$

Simplify,

$$x^2 - 4kx + (2 + 2k)$$

Earlier we deduced that $b^2 - 4ac > 0$,

$$a = 1, b = -4k, c = 2 + 2k$$

Substitute these values into $b^2 - 4ac > 0$,

$$b^2 - 4ac > 0$$

$$(-4k)^2 - 4(1)(2 + 2k) > 0$$

Expand the brackets,

$$16k^2 - 8 - 8k > 0$$

Rearrange into the form $ax^2 + bx + c$,

$$16k^2 - 8k - 8 > 0$$

Divide through by 8,

$$2k^2 - k - 1 > 0$$

Solve the quadratic inequality,

$$(2k + 1)(k - 1) = 0$$

$$2k + 1 = 0 \qquad\qquad k - 1 = 0$$

$$k = -\frac{1}{2} \qquad\qquad k = 1$$

$$k < -\frac{1}{2} \text{ and } k > 1$$

Therefore, the final answer is,

$$k < -\frac{1}{2} \text{ and } k > 1$$