GCE A Level Maths 9709

SMIYL

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1.6 Series

In this topic we will learn how to:

- use the formula for the nth term, sum of first n terms and sum to infinity to solve problems involving both arithmetic and geometric progressions

Arithmetic and Geometric Progressions

To solve problems involving both arithmetic and geometric progressions, you have to use your knowledge of the two progressions. Let's learn how to solve questions of the sort, by walking through some past paper questions.

1. The first term of an arithmetic progression is a and the common difference is -4. The first term of a geometric progression is $5a$ and the common ratio is $-\frac{1}{4}$. The sum to infinity of the geometric progression is equal to the sum of the first eight terms of the arithmetic progression. Find the value of a. (9709/11/O/N/21 number 4)

   Let's write out all the information we have been given,

   Arithmetic Progression

   $$u_1 = a \qquad d = -4$$

   Geometric Progression

   $$u_1 = 5a \qquad r = -\frac{1}{4}$$

   $$S_\infty = S_{8\ AP}$$

Now let's use the formulae for sum to infinity and sum of first n terms of an arithmetic progression,

$$S_\infty = S_{8\ AP}$$

$$\frac{a}{1-r} = \frac{1}{2}n(2a + (n-1)d)$$

Substitute into the formulae,

$$\frac{5a}{1 - \left(-\frac{1}{4}\right)} = \frac{1}{2}(8)(2a + (8-1)(-4))$$

Simplify and solve for a,

$$\frac{5a}{\frac{5}{4}} = 4(2a - 28)$$

$$4a = 8a - 112$$

$$8a - 4a = 112$$

$$4a = 112$$

$$a = 28$$

Therefore, the final answer is,

$$a = 28$$

2. The first term of a geometric progression and the first term of an arithmetic progression are both equal to a. The third term of a geometric progression is equal to the second term of the arithmetic progression. The fifth term of the geometric progression is equal to the sixth term of the arithmetic progression. Given that the terms are all positive and not all equal, find the sum of the first twenty terms of the arithmetic progression in terms of a. (9709/12/F/M/22 number 4)

   Let's write out the information we have been given,

$$u_{1\ GP} = u_{1\ AP} \qquad u_{3\ GP} = u_{2\ AP} \qquad u_{5\ GP} = u_{6\ AP}$$

   Using the formula for the nth term in the respective progressions, the above can be written as,

$$a = a \qquad ar^2 = a + d \qquad ar^4 = a + 5d$$

We can use the second and third equations, to solve simultaneously for d,

$$ar^2 = a + d \qquad ar^4 = a + 5d$$

Make r^2 the subject of the formula in the first equation,

$$ar^2 = a + d$$

$$r^2 = \frac{a+d}{a}$$

Substitute r^2 into the second equation,

$$ar^4 = a + 5d$$

$$a\left(r^2\right)^2 = a + 5d$$

$$a\left(\frac{a+d}{a}\right)^2 = a + 5d$$

Expand the quadratic,

$$a\frac{(a+d)^2}{a^2} = a + 5d$$

Cancel out the a's

$$\frac{(a+d)^2}{a} = a + 5d$$

Expand the quadratic,

$$\frac{a^2 + 2ad + d^2}{a} = a + 5d$$

Get rid of the denominator,

$$a^2 + 2ad + d^2 = a(a + 5d)$$

Expand the parentheses on the right-hand side,

$$a^2 + 2ad + d^2 = a^2 + 5ad$$

Put all the terms on one side,

$$a^2 - a^2 + 5ad - 2ad - d^2 = 0$$

Simplify,

$$3ad - d^2 = 0$$

Factor out the d,

$$d(3a - d) = 0$$

Solve for d separately, in terms of a,

$$d = 0 \qquad 3a - d = 0$$

$$d = 0 \qquad d = 3a$$

$$d = 3a$$

Note: We disregard $d = 0$, because the question states that the terms are not all equal. If $d = 0$ then all the terms will be equal.

Now that we have d in terms of a we can find the sum of the first twenty terms of the arithmetic progression in terms of a,

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

Substitute,

$$S_{20} = \frac{1}{2}(20)(2a + (20-1)(3a))$$

Simplify and find S_{20} in terms of a,

$$S_{20} = 10(2a + 57a)$$

$$S_{20} = 10(59a)$$

$$S_{20} = 590a$$

Therefore, the final answer is,

$$S_{20} = 590a$$

3. The first term of a geometric progression is 216 and the common ratio is $\frac{2}{3}$. The second term of the geometric progression is equal to the second term of an arithmetic progression. The third term of the geometric progression is equal to the fifth term of the same arithmetic progression. Find the sum of the first 21 terms of the arithmetic progression. (9709/13/O/N/22 number 9)

Let's write out the information we have been given,

Geometric Progression

$$a = 216 \qquad r = \frac{2}{3}$$

$$u_{2\ GP} = u_{2\ AP} \qquad u_{3\ GP} = u_{5\ AP}$$

Using the formula for the nth **term in the respective progressions, the above can be written as,**

$$ar = a + d \qquad ar^2 = a + 4d$$

Substitute a **and** r **for the geometric progression parts,**

$$216\left(\frac{2}{3}\right) = a + d \qquad 216\left(\frac{2}{3}\right)^2 = a + 4d$$

$$144 = a + d \qquad 96 = a + 4d$$

Solve the two equations simultaneously for a **and** d**,**

$$144 = a + d \qquad 96 = a + 4d$$

Make a **the subject of the formula in the first equation,**

$$a = 144 - d$$

Substitute a **in the second equation and solve for** d**,**

$$96 = a + 4d$$

$$96 = 144 - d + 4d$$

$$96 = 144 + 3d$$

$$3d = 96 - 144$$

$$3d = -48$$

$$d = -16$$

Substitute d **to find** a**,**

$$a = 144 - d$$

$$a = 144 - (-16)$$

$$a = 160$$

Now that we have a and d, we can find the sum of the first 21 terms of the arithmetic progression,

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

Substitute,

$$S_{21} = \frac{1}{2}(21)(2(160) + (21-1)(-16))$$
$$S_{21} = 0$$

Therefore, the final answer is,

$$S_{21} = 0$$