GCE A Level Maths 9709

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1.5 Trigonometry

In this topic we will learn how to:

- use the identities $\frac{\sin\theta}{\cos\theta} \equiv \tan\theta$ and $\sin^2\theta + \cos^2\theta \equiv 1$

Pythagorean Identities

The two trig identities displayed below will be used in proving other trig identities,

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

In some cases, the second trig identity, may be useful in the forms below,

$$\sin^2\theta \equiv 1 - \cos^2\theta$$

$$\cos^2\theta \equiv 1 - \sin^2\theta$$

General tips for solving trig identities,

- Work from the more complex side to the less complex side
- Reduce anything in terms of $\tan x$ to be in terms of $\sin x$ and $\cos x$
- Combine any separate fractions into one single fraction
- Expose yourself to a lot trig identity questions

Let's look at some past paper questions.

1. Show that,

$$\frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \frac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta} \equiv \frac{4}{5\cos^2\theta - 4}$$

(9709/12/F/M/22 number 7)

We will work from the left-hand side to the right-hand side,

$$\frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \frac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta}$$

Compose the two fractions into one fraction,

$$\frac{(\sin\theta + 2\cos\theta)(\cos\theta + 2\sin\theta) - (\sin\theta - 2\cos\theta)(\cos\theta - 2\sin\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Expand the numerator,

$$\frac{\sin\theta\cos\theta + 2\sin^2\theta + 2\cos^2\theta + 4\sin\theta\cos\theta - (-2\sin^2\theta + \sin\theta\cos\theta - 2\cos^2\theta + 4\sin\theta\cos\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Simplify the numerator,

$$\frac{5\sin\theta\cos\theta + 2\sin^2\theta + 2\cos^2\theta - (-2\sin^2\theta - 2\cos^2\theta + 5\sin\theta\cos\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

$$\frac{5\sin\theta\cos\theta + 2(\sin^2\theta + \cos^2\theta) - (-2(\sin^2\theta + 2\cos^2\theta) + 5\sin\theta\cos\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Use the identity $\sin^2 x + \cos^2 x \equiv 1$**,**

$$\frac{5\sin\theta\cos\theta + 2(1) - (-2(1) + 5\sin\theta\cos\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

$$\frac{5\sin\theta\cos\theta + 2 - (-2 + 5\sin\theta\cos\theta)}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Finish expanding the numerator,

$$\frac{5\sin\theta\cos\theta + 2 + 2 - 5\sin\theta\cos\theta}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Finish simplifying the numerator,

$$\frac{4}{(\cos\theta - 2\sin\theta)(\cos\theta + 2\sin\theta)}$$

Expand the denominator,

$$\frac{4}{\cos^2\theta + 2\sin\theta\cos\theta - 2\sin\theta\cos\theta - 4\sin^2\theta}$$

Simplify the denominator,

$$\frac{4}{\cos^2\theta - 4\sin^2\theta}$$

Use the identity $\sin^2 x \equiv 1 - \cos^2 x$ **in the denominator,**

$$\frac{4}{\cos^2\theta - 4(1-\cos^2\theta)}$$

Simplify the denominator,

$$\frac{4}{\cos^2\theta - 4 + 4\cos^2\theta}$$
$$\frac{4}{\cos^2\theta + 4\cos^2\theta - 4}$$
$$\frac{4}{5\cos^2\theta - 4}$$

Therefore,

$$\frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} - \frac{\sin\theta - 2\cos\theta}{\cos\theta + 2\sin\theta} \equiv \frac{4}{5\cos^2\theta - 4}$$

2. Prove the identity,

$$\frac{1+\sin x}{1-\sin x} - \frac{1-\sin x}{1+\sin x} \equiv \frac{4\tan x}{\cos x}$$

(9709/12/M/J/21 number 10)

We will work from the left-hand side to the right-hand side,

$$\frac{1+\sin x}{1-\sin x} - \frac{1-\sin x}{1+\sin x}$$

Compose the two fractions into one fraction,

$$\frac{(1+\sin x)^2 - (1-\sin x)^2}{(1-\sin x)(1+\sin x)}$$

Expand the numerator,

$$\frac{1 + 2\sin x + \sin^2 x - (1 - 2\sin x + \sin^2 x)}{(1-\sin x)(1+\sin x)}$$

$$\frac{1 + 2\sin x + \sin^2 x - 1 + 2\sin x - \sin^2 x}{(1-\sin x)(1+\sin x)}$$

Group like terms and simplify the numerator,

$$\frac{1 - 1 + 2\sin x + 2\sin x + \sin^2 x - \sin^2 x}{(1 - \sin x)(1 + \sin x)}$$

$$\frac{4\sin x}{(1 - \sin x)(1 + \sin x)}$$

Expand the denominator,

$$\frac{4\sin x}{1 - \sin^2 x}$$

Use the identity $\cos^2 x \equiv 1 - \sin^2 x$ **in the denominator,**

$$\frac{4\sin x}{\cos^2 x}$$

Simplify,

$$\frac{4\sin x}{\cos x \times \cos x}$$

$$\frac{4\sin x}{\cos x} \times \frac{1}{\cos x}$$

$$4\tan x \times \frac{1}{\cos x}$$

$$\frac{4\tan x}{\cos x}$$

Therefore,

$$\frac{1 + \sin x}{1 - \sin x} - \frac{1 - \sin x}{1 + \sin x} \equiv \frac{4\tan x}{\cos x}$$

3. Prove the identity,

$$\frac{1 - 2\sin^2 \theta}{1 - \sin^2 \theta} = 1 - \tan^2 \theta$$

(9709/11/M/J/21 number 7)

We will work from the right-hand side to the left-hand side,

$$1 - \tan^2 \theta$$

Note: In this case, you could just as easily work from left-hand side to right-hand side, however, I chose the opposite because it is easier to simplify $\tan^2 \theta$ **to be in terms of** $\sin \theta$ **and** $\cos \theta$**.**

Use the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$**,**

$$1 - \frac{\sin^2\theta}{\cos^2\theta}$$

Compose the two terms into one fraction,

$$\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

Use the identity $cos^2 \equiv 1 - \sin^2\theta$ **in the numerator,**

$$\frac{1 - \sin^2\theta - \sin^2\theta}{\cos^2\theta}$$

Simplify the numerator,

$$\frac{1 - 2\sin^2\theta}{\cos^2\theta}$$

Use the identity $cos^2 \equiv 1 - \sin^2\theta$ **in the denominator,**

$$\frac{1 - 2\sin^2\theta}{1 - \sin^2\theta}$$

Therefore,

$$\frac{1 - 2\sin^2\theta}{1 - \sin^2\theta} = 1 - \tan^2\theta$$

4. Prove the identity,

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1 + \sin\theta} \equiv -\tan^2\theta(1 + \sin^2\theta)$$

(9709/11/M/J/22 number 4)

We will work from the left-hand side to the right-hand side,

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1 + \sin\theta}$$

Compose the two fractions into one fraction,

$$\frac{\sin^3\theta(1 + \sin\theta) - \sin^2\theta(\sin\theta - 1)}{(\sin\theta - 1)(1 + \sin\theta)}$$

Expand the numerator and simplify,

$$\frac{\sin^3\theta + \sin^4\theta - \sin^3\theta + \sin^2\theta}{(\sin\theta - 1)(1 + \sin\theta)}$$

$$\frac{\sin^4\theta + \sin^2\theta}{(\sin\theta - 1)(1 + \sin\theta)}$$

Factor out $\sin^2\theta$ **in the numerator,**

$$\frac{\sin^2\theta(\sin^2\theta + 1)}{(\sin\theta - 1)(1 + \sin\theta)}$$

Expand the denominator,

$$\frac{\sin^2\theta(1 + \sin^2\theta)}{\sin\theta + \sin^2\theta - 1 - \sin\theta}$$

Group like terms and simplify in the denominator,

$$\frac{\sin^2\theta(1 + \sin^2\theta)}{\sin\theta - \sin\theta + \sin^2\theta - 1}$$

$$\frac{\sin^2\theta(1 + \sin^2\theta)}{\sin^2\theta - 1}$$

Use the identity $\sin^2\theta \equiv 1 - \cos^2\theta$ **in the denominator,**

$$\frac{\sin^2\theta(1 + \sin^2\theta)}{1 - \cos^2\theta - 1}$$

Simplify the denominator,

$$\frac{\sin^2\theta(1 + \sin^2\theta)}{-\cos^2\theta}$$

Simplify,

$$-\tan^2\theta(1 + \sin^2\theta)$$

Therefore,

$$\frac{\sin^3\theta}{\sin\theta - 1} - \frac{\sin^2\theta}{1 + \sin\theta} \equiv -\tan^2\theta(1 + \sin^2\theta)$$