GCE A Level Maths 9709

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1.5 Trigonometry

In this topic we will learn how to:

- find all the solutions of simple trigonometrical equations lying in a specified interval

Solving Trig Equations

To solve trig equations, you have to reduce them to be in terms of one trig function. Once you have achieved that, use one of the formulae below, depending on which trig function you have, to find the all the solutions within the specified interval. For $\sin x$ **use the formulae,**

$$P.V(-1)^n + 180n$$

$$P.V(-1)^n + \pi n$$

Where $P.V$ **is the Principal Value and** n **is an integer.**

Note: Use the relevant formula depending on whether the solutions should be in degrees or radians.

For $\cos x$ **use the formulae,**

$$\pm P.V + 360n$$

$$\pm P.V + 2\pi n$$

Where $P.V$ **is the Principal Value and** n **is an integer.**

For $\tan x$ **use the formulae,**

$$P.V + 180n$$

$$P.V + \pi n$$

Where $P.V$ is the Principal Value and n is an integer.

Let's apply these formulae to past paper questions.

1. Solve by factorising, the equation

$$6\cos\theta\tan\theta - 3\cos\theta + 4\tan\theta - 2 = 0$$

for $0° \leq \theta \leq 360°$. (9709/11/O/N/21 number 3)

$$6\cos\theta\tan\theta - 3\cos\theta + 4\tan\theta - 2 = 0$$

We will factorise by grouping,

$$6\cos\theta\tan\theta - 3\cos\theta + 4\tan\theta - 2 = 0$$

Factor out $3\cos\theta$ in the first two terms

$$3\cos\theta(2\tan\theta - 1) + 4\tan\theta - 2 = 0$$

Factor out 2 in the last two terms,

$$3\cos\theta(2\tan\theta - 1) + 2(2\tan\theta - 1) = 0$$

Factor out $2\tan\theta - 1$ since it is common,

$$(3\cos\theta + 2)(2\tan\theta - 1) = 0$$

Solve for θ,

$$\begin{aligned} 3\cos\theta + 2 &= 0 & \qquad 2\tan\theta - 1 &= 0 \\ 3\cos\theta &= -2 & 2\tan\theta &= 1 \\ \cos\theta &= \frac{-2}{3} & \tan\theta &= \frac{1}{2} \\ \theta &= \cos^{-1}\left(\frac{-2}{3}\right) & \theta &= \tan^{-1}\left(\frac{1}{2}\right) \\ P.V &= 131.8103149 & P.V &= 26.56505118 \end{aligned}$$

Use the formulae above to check if there are any other solutions within the specified interval,

$$\begin{aligned} P.V &= 131.8103149 & \qquad P.V &= 26.56505118 \\ \theta &= \cos^{-1}\left(\frac{-2}{3}\right) & \theta &= \tan^{-1}\left(\frac{1}{2}\right) \\ &\pm P.V + 360n & & P.V + 180n \end{aligned}$$

At $n = 1$,

$$P.V + 360(1) = 491.81 \qquad P.V + 180(1) = 206.57$$
$$-P.V + 360(1) = -311.81$$

All the solutions at $n = 1$ are out of range, so we do not consider them, therefore the final answer is,

$$\theta = 26.6°, 131.8°$$

Note: $P.V$ counts as a solution. It is essentially the solution at $n = 0$.

2. Solve the equation
$$\frac{\tan\theta + 2\sin\theta}{\tan\theta - 2\sin\theta} = 3$$
for $0° < \theta < 180°$. (9709/12/F/M/21 number 3)

$$\frac{\tan\theta + 2\sin\theta}{\tan\theta - 2\sin\theta} = 3$$

Start by getting rid of the denominator,

$$\tan\theta + 2\sin\theta = 3(\tan\theta - 2\sin\theta)$$

Expand the right-hand side,

$$\tan\theta + 2\sin\theta = 3\tan\theta - 6\sin\theta$$

Put all the terms on one side,

$$3\tan\theta - \tan\theta - 6\sin\theta - 2\sin\theta = 0$$

Simplify,

$$2\tan\theta - 8\sin\theta = 0$$

Use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ to replace $\tan\theta$,

$$2\left(\frac{\sin\theta}{\cos\theta}\right) - 8\sin\theta = 0$$
$$\frac{2\sin\theta}{\cos\theta} - 8\sin\theta = 0$$

Factor out $2\sin\theta$,

$$2\sin\theta\left(\frac{1}{\cos\theta}-4\right)=0$$

Solve for θ separately,

$$\begin{array}{cc}
2\sin\theta=0 & \dfrac{1}{\cos\theta}-4=0 \\
\sin\theta=0 & \dfrac{1}{\cos\theta}=4 \\
\sin\theta=0 & 4\cos\theta=1 \\
\sin\theta=0 & \cos\theta=\dfrac{1}{4} \\
\theta=\sin^{-1}(0) & \theta=\cos^{-1}\left(\dfrac{1}{4}\right) \\
P.V=0 & P.V=75.52248781
\end{array}$$

Use the formulae to check if there are any other solutions within range,

$$\begin{array}{cc}
P.V=0 & P.V=75.52248781 \\
P.V(-1)^n+180n & \pm P.V+360n
\end{array}$$

At $n=1$,

$$P.V(-1)^1+180(1)=180 \qquad \pm P.V+360(1)=(284.5,435.5)$$

All the solutions at $n=1$ are out of range, so we do not consider them, therefore, the final ansewr is,

$$\theta=75.5^\circ$$

Note: If the angle is in degrees, give your answer correct to 1 decimal place. If the angle is in radians, give your answer correct to 3 significant figures.

3. Solve the equation

$$2\cos\theta=7-\frac{3}{\cos\theta}$$

for $-90^\circ<\theta<90^\circ$. (9709/12/O/N/21 number 1)

$$2\cos\theta = 7 - \frac{3}{\cos\theta}$$

Get rid of the denominator,

$$2\cos^2\theta = 7\cos\theta - 3$$

Put all the terms on one side,

$$2\cos^2\theta - 7\cos\theta + 3 = 0$$

Factorise the quadratic,

$$2\cos^2\theta - 7\cos\theta + 3 = 0$$

$$(2\cos\theta - 1)(\cos\theta - 3) = 0$$

Solve for θ separately,

$$2\cos\theta - 1 = 0 \qquad \cos\theta - 3 = 0$$

$$2\cos\theta = 1 \qquad \cos\theta = 3$$

$$\cos\theta = \frac{1}{2} \qquad \cos\theta = 3$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right) \qquad \theta = \cos^{-1}(3)$$

$$\theta = 60 \qquad \theta = nosolutions$$

Note: The graph of $y = \cos x$ ranges form -1 to 1, so 3 is out of range, hence no solution for $\theta = \cos^{-1}(3)$.

$$P.V = 60$$

Use the formulae to check if there any other solutions within range,

$$\pm P.V + 360n$$

At $n = 0$,

$$\pm P.V + 360(0) = \pm 60$$

At $n = 1$,

$$\pm P.V + 360(1) = (-300, 420)$$

The solutions at $n = 1$ are out of range so disregard them. Therefore, the final answer is,

$$\theta = -60^\circ, 60^\circ$$

4. Solve the equation

$$\frac{4}{5\cos^2\theta - 4} = 5$$

(9709/12/F/M/22 number 7)

$$\frac{4}{5\cos^2\theta - 4} = 5$$

Get rid of the denominator,

$$4 = 5(5\cos^2\theta - 4)$$

Expand the right-hand side,

$$4 = 25\cos^2\theta - 20$$

Simplify,

$$25\cos^2\theta = 20 + 4$$
$$25\cos^2\theta = 24$$
$$\cos^2\theta = \frac{24}{25}$$

Take the square root of both sides,

$$\cos^2\theta = \frac{24}{25}$$
$$\sqrt{\cos^2\theta} = \pm\sqrt{\frac{24}{25}}$$
$$\cos\theta = \pm\sqrt{\frac{24}{25}}$$

Solve for θ separately,

$$\cos\theta = -\sqrt{\frac{24}{25}} \qquad \cos\theta = \sqrt{\frac{24}{25}}$$
$$\theta = \cos^{-1}\left(-\sqrt{\frac{24}{25}}\right) \qquad \theta = \cos^{-1}\left(\sqrt{\frac{24}{25}}\right)$$
$$P.V = 168.463041 \qquad P.V = 11.53695903$$

Use the formulae to check if there are any other solutions within range,

$$P.V = 168.463041 \qquad\qquad P.V = 11.53695903$$

$$\pm P.V + 360n \qquad\qquad \pm\, P.V + 360n$$

At $n = 1$,

$$\pm P.V + 360(1) = 528.5 \qquad\qquad \pm\, P.V + 360(1) = 371.5$$

The solutions at $n = 1$ are out of range, so we do not consider them. Therefore, the final answer is,

$$\theta = 11.5^\circ, 168.5^\circ$$