GCE A Level Maths 9709

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1.2 Functions

In this topic we will learn how to:

- understand the terms function, domain, range, one-one function, inverse function and composition of functions
- identify the range of a given function in simple cases, and find the composition of two given functions

Definitions and Range of a Function

Key Terms

A function is an expression that defines the relationship between one variable and another variable.

The domain of a function represents all the input values it can take. They are typically denoted as x values.

The range of a function represents all the output values. They are typically denoted as y values.

A one-one function is one that maps every distinct input value to exactly one distinct output value. One x value maps onto one y value. To check if a function is a one-one, you can use the horizontal line test. Sketch the function on a graph, if a horizontal line is drawn at any point, it should only pass through the graph once. If at any point, it passes through the graph more than once, it is not a one-one function. If at every point, it passes through only once, it is a one-one function.

An inverse function is one that reverses the operations of the original function. It is also called an anti function.

A composite funciton is a function made out of other functions, where its input is the output of another function.

Range of a Function

To determine the range of a function, we have to consider the type of function it is. If the function is linear, substitute the extreme values of the domain into the function, these will give you the range.

If a function is quadratic, find its vertex by completing the square. Use that to sketch the graph of the function. From the graph, you can determine the range.

If a function is rational, to find its range, we will use trial and error OR the idea of ∞ as the upper limit of the domain. See the second question below.

Let's look at some past paper questions.

1. The function f is defined by $f(x) = 2x^2 - 16x + 23$ for $x < 3$. Find the range of f. (9709/13/M/J/22 number 6)

$$f(x) = 2x^2 - 16x + 23$$

To find the range of a quadratic we first need to find the vertex. To find the vertex, we have to complete the square,

$$y = 2x^2 - 16x + 23$$

Complete the square using your preferred method,

$$y = 2\left(x^2 - 8x\right) + 23$$

$$y = 2\left[(x-4)^2 - 16\right] + 23$$

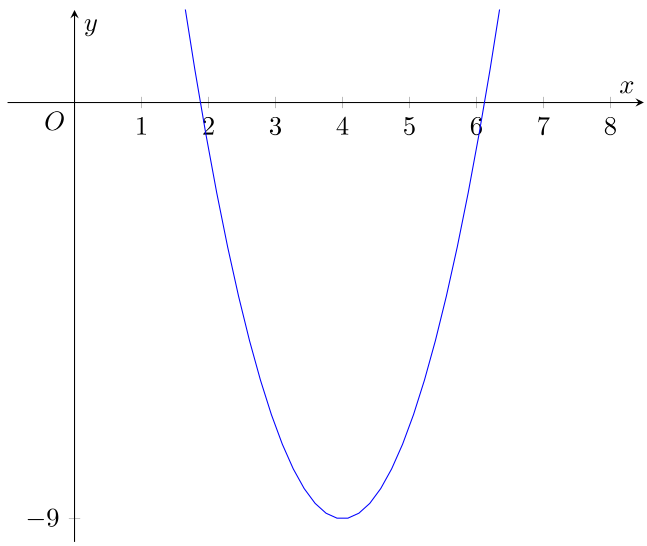
$$y = 2(x-4)^2 - 32 + 23$$

$$y = 2(x-4)^2 - 9$$

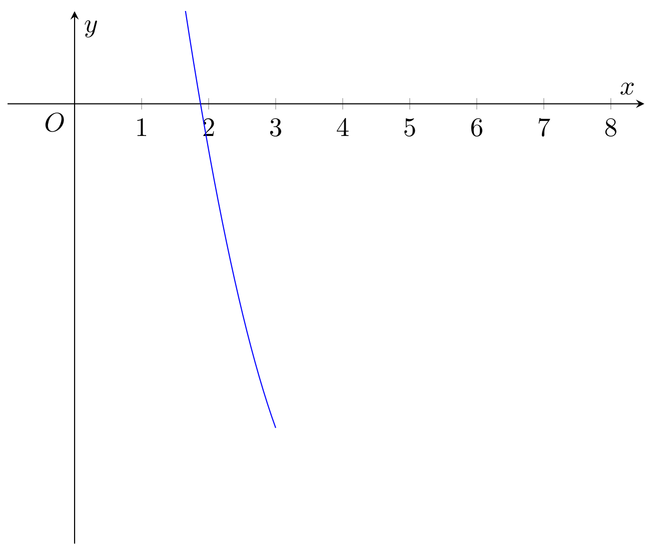
Therefore, f can be written as,

$$f(x) = 2(x-4)^2 - 9$$

The vertex of f is $(4, -9)$, let's use that to sketch the graph of f,



Our domain is $x < 3$,



To find the minimum point of the function, substitute $x = 3$ into f,

$$f(x) = 2(x-4)^2 - 9$$
$$f(3) = 2(3-4)^2 - 9$$
$$f(3) = -7$$

Therefore, the range of f is,

$$f(x) > -7$$

2. Functions f and g are defined by

$$f : x \mapsto \frac{3}{2x+1} \quad \text{for } x > 0$$

$$g : x \mapsto \frac{1}{x} + 2 \quad \text{for } x > 0$$

Find the range of f and the range of g. (9709/11/O/N/19 number 7)

Let's start with f,

$$f(x) = \frac{3}{2x+1} \quad \text{for } x > 0$$

Note: The notation $f(x) =$ is the same as $f : x \mapsto$.

f is a rational function, for the domain $x > 0$. The domain $x > 0$ can be rewritten as,

$$0 < x < \infty$$

Note: If x is greater than 0 then the domain runs from 0 to ∞.

Substitute the extreme values of the domain into the function. Let's start with $x = 0$,

$$f(x) = \frac{3}{2x+1}$$

$$f(0) = \frac{3}{2(0)+1}$$

$$f(0) = 3$$

Let's substitute in $x = \infty$,

$$f(x) = \frac{3}{2x+1}$$

$$f(\infty) = \frac{3}{2(\infty)+1}$$

$$f(\infty) = 0$$

Note: $\frac{a}{\infty} = 0$, where a is a small number. Alternatively, you can substitute values of x into the function, you will notice that as x gets larger, f approaches 0.

Therefore, the range of f is,

$$0 < f(x) < 3$$

Let's move on to g,

$$g(x) = \frac{1}{x} + 2 \quad \text{for } x > 0$$

g is a rational function, for the domain $x > 0$. The domain $x > 0$ can be rewritten as,

$$0 < x < \infty$$

Let's start by writing g as one fraction,

$$g(x) = \frac{1}{x} + 2$$

$$g(x) = \frac{1 + 2x}{x}$$

Substitute the extreme values of the domain into the function. Let's start with $x = 0$,

$$g(x) = \frac{1 + 2x}{x}$$

$$g(0) = \frac{1 + 2(0)}{0}$$

$$g(0) = \frac{1}{0}$$

$$g(0) = \infty$$

Note: $\frac{a}{0} = \infty$ where a is a small number.

Let's substitute in $x = \infty$,

$$g(x) = \frac{1 + 2x}{x}$$

$$g(\infty) = \frac{1 + 2(\infty)}{\infty}$$

$$g(\infty) = \frac{2(\infty)}{\infty}$$

Note: Compared to ∞, 1 is very small so we can ignore it.

Cancel out ∞,

$$g(\infty) = 2$$

Note: If the whole concept of ∞ confuses you, you can use trial and error instead. Substitute values of x that satisfy the domain. In this case, you will notice that as x increases g approaches 2 but never reaches it.

The domain of g is,

$$2 < g(x) < \infty$$

Note: If you're familiar with the graph of $y = \frac{1}{x}$ you can consider g is a transformation of that graph and use that to find the range of g.

This can be written as,

$$g(x) > 2$$

Therefore, the range of g is,

$$g(x) > 2$$

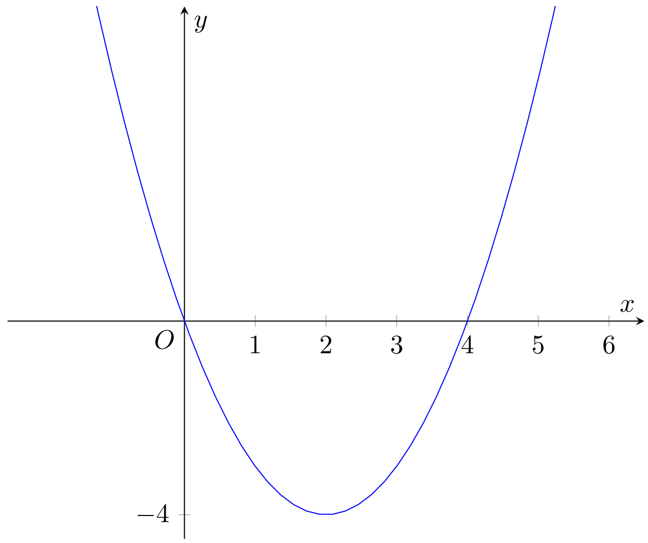
Therefore, the final answer is,

$$0 < f(x) < 3 \qquad\qquad g(x) > 2$$

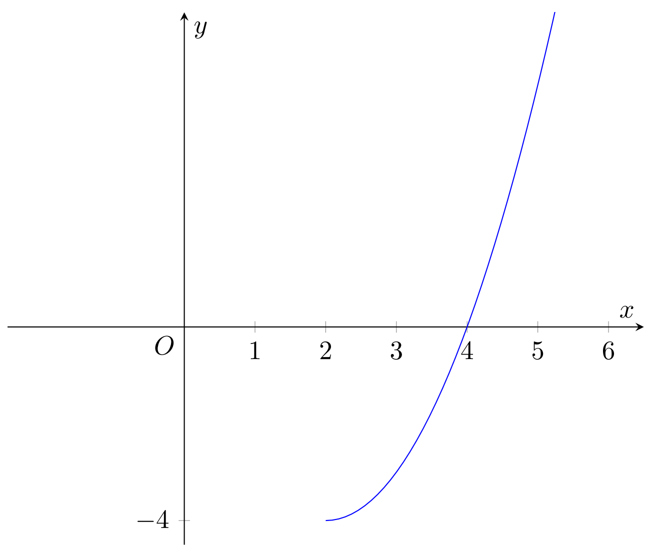
3. The function f is defined by $f(x) = (x-2)^2 - 4$ for $x \geq 2$. State the range of f. (9709/11/M/J/21 number 9)

$$f(x) = (x-2)^2 - 4 \text{ for } x \geq 2$$

The square has already been completed, so we can use the vertex $(2, -4)$ to sketch f,



The domain is $x \geq 2$,



Since the turning point of f is $(2, -4)$, we know that the minimum value of f is -4. Alternatively, you can still calculate it.

Substitute the $x = 2$ into f,

$$f(x) = (x-2)^2 - 4$$
$$f(2) = (2-2)^2 - 4$$
$$f(2) = -4$$

Therefore, the final answer is,

$$f(x) \geq -4$$

Note: We use the $\geq$ symbol in the range because the domain is also defined with the $\geq$ symbol.