

GCE A Level Maths 9709

SMIYL

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1.2 Functions

In this topic we will learn how to:

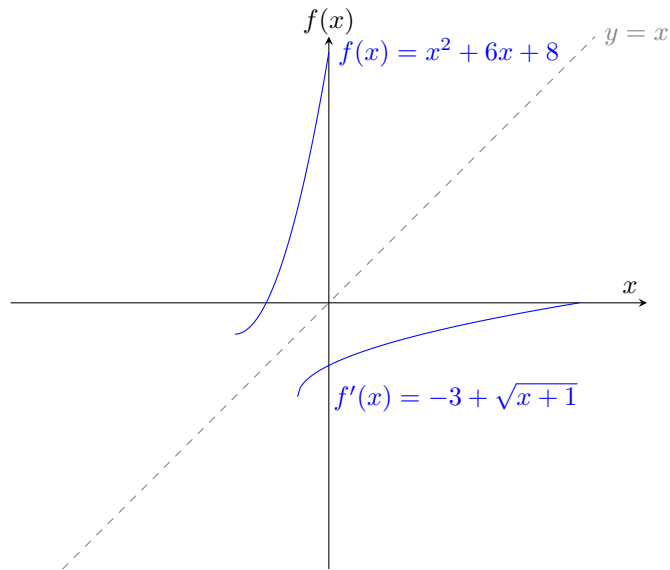
- Find the inverse of a one-one function in simple cases

Inverse of a Function

The inverse or anti function reverses the operations of the original function. For the function $f(x)$ the inverse is denoted by,

$$f^{-1}(x)$$

To find the inverse of a function, make x the subject of the formula in the original function. The resulting function, is the inverse. The inverse is a reflection of the original function in the line $y = x$. As a result, the domain of the original function is the range of the inverse. The range of the original function is the domain of the inverse. However, remember to always write the domain in terms of x and the range in terms of y .



An example of the graphical relationship between a function and its inverse.

Let's walk through some examples to understand the process.

1. The function f is defined by $f(x) = \frac{2x}{3x-1}$ for $x > \frac{1}{3}$. Find an expression for $f^{-1}(x)$. (9709/13/O/N/20 number 6)

$$f(x) = \frac{2x}{3x-1}$$

Substitute $f(x)$ with y ,

$$y = \frac{2x}{3x-1}$$

The next step is to make x the subject of the formula. We will do this in steps. Start by multiplying both sides by the denominator,

$$y(3x-1) = 2x$$

Expand the bracket,

$$3xy - y = 2x$$

Put all terms containing x on one side,

$$3xy - 2x = y$$

Factor out x ,

$$x(3y - 2) = y$$

Divide both sides by the term in brackets, $3y - 2$,

$$x = \frac{y}{3y - 2}$$

Rewrite the function using the notation for an inverse function, substituting y with x ,

$$x = \frac{y}{3y - 2}$$
$$f^{-1}(x) = \frac{x}{3x - 2}$$

Therefore, the final answer is,

$$f^{-1}(x) = \frac{x}{3x - 2}$$

2. The function f is defined by $f(x) = x^2 - 4x + 7$ for $x < 1$. Find an expression for $f^{-1}(x)$ and state the domain of f^{-1} . (9709/12/F/M/19 number 8)

$$f(x) = x^2 - 4x + 7$$

Substitute $f(x)$ with y ,

$$y = x^2 - 4x + 7$$

Complete the square,

$$y = (x - 2)^2 + 3$$

Make x the subject of the formula. Start by subtracting 3 from both sides,

$$y - 3 = (x - 2)^2$$

Take the square root of both sides,

$$\pm\sqrt{y - 3} = \sqrt{(x - 2)^2}$$

Simplify,

$$\pm\sqrt{y - 3} = x - 2$$

Add 2 to both sides,

$$x = 2 \pm \sqrt{y - 3}$$

Determine which sign the inverse should take,

$$x = 2 - \sqrt{y - 3}$$

Note: You can use trial and error to see which sign satisfies the inverse. A general rule of thumb is that if the original function has a domain of the form $x < k$, the inverse takes a negative sign. If it has a domain of the form $x > k$, the inverse takes a positive sign.

Rewrite the function using notation for an inverse function, substituting y with x ,

$$f^{-1}(x) = 2 - \sqrt{x - 3}$$

The domain of the inverse is the range of the original function, therefore let's find the range of the original function,

$$f(x) = x^2 - 4x + 7 \text{ for } x < 1$$

According to the domain the maximum x -value of the function is 1. Let's substitute that into the function to find its minimum value,

$$f(1) = 1^2 - 4(1) + 7$$

$$f(1) = 4$$

It's minimum value is 4.

Therefore, the range of $f(x)$ is,

$$y > 4$$

Now that we have the range of $f(x)$ we can determine the domain of $f^{-1}(x)$. The domain of $f^{-1}(x)$ is,

$$x > 4$$

Therefore, the final answer is,

$$f^{-1}(x) = 2 - \sqrt{x - 3} \text{ for } x > 4$$

3. Functions f and g are defined by

$$f(x) = 4x - 2 \text{ for } x \in \mathbb{R}$$

$$g(x) = \frac{4}{x+1} \text{ for } x \in \mathbb{R}, x \neq -1.$$

Find the values of x for which $f^{-1}(x) = g^{-1}(x)$. (9709/12/O/N/20 number 5)

First let's find $f^{-1}(x)$,

$$f(x) = 4x - 2$$

Substitute $f(x)$ with y ,

$$y = 4x - 2$$

Make x the subject of the formula,

$$y = 4x - 2$$

Add 2 to both sides,

$$4x = y + 2$$

Divide both sides by 4,

$$x = \frac{y + 2}{4}$$

Rewrite the function using notation for an inverse function, substituting y with x ,

$$f^{-1}(x) = \frac{x + 2}{4}$$

Now let's find $g^{-1}(x)$,

$$g(x) = \frac{4}{x+1}$$

Substitute $g(x)$ with y ,

$$y = \frac{4}{x+1}$$

Make x the subject of the formula,

$$y = \frac{4}{x+1}$$

Multiply both sides by the denominator, $x + 1$,

$$y(x+1) = 4$$

Expand the bracket on the left-hand side,

$$xy + y = 4$$

Move y to the right-hand side,

$$xy = 4 - y$$

Divide both sides by y ,

$$x = \frac{4 - y}{y}$$

$$x = \frac{4}{y} - 1$$

Rewrite the function using notation for an inverse function, substituting y with x ,

$$g^{-1}(x) = \frac{4}{x} - 1$$

Now let's solve the question,

$$f^{-1}(x) = g^{-1}(x)$$

$$\frac{x+2}{4} = \frac{4}{x} - 1$$

Multiply both sides by 4,

$$x+2 = 4\left(\frac{4}{x} - 1\right)$$

Expand the bracket on the right-hand side,

$$x+2 = \frac{16}{x} - 4$$

Group like terms,

$$x + 2 + 4 = \frac{16}{x}$$

Simplify

$$x + 6 = \frac{16}{x}$$

Mutlply through by x ,

$$x^2 + 6x = 16$$

Put all the terms on one side and solve the quadratic using your preferred method. In this example, we will factorise,

$$x^2 + 6x - 16 = 0$$

$$(x + 8)(x - 2) = 0$$

$$x = -8 \quad x = 2$$

Therefore, the final answer is,

$$x = -8 \quad x = 2$$