GCE A Level Maths 9709

SMIYL

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1.6 Series

In this topic we will learn how to:

- recognise geometric progressions
- use the formulae for the nth term and for the sum of the first n terms to solve problems involving geometric progressions
- use the condition for convergence of a geometric progression, and the formula for the sum to infinity of a convergent geometric progression

Geometric Progressions

A geometric progression is a sequence of numbers in which each differs from the preceding one by a common ratio. To find the nth term under a geometric progression, we use the formula,

$$u_n = ar^{n-1}$$

Where u_n represents the nth term, a represents the first term of the progression, n represents the position of the nth term and r represents the common ratio.

To find the sum of the first n terms under a geometric progression, we use the following formula,

$$S_n = \frac{a(1-r^n)}{1-r}$$

Where S_n represents sum of first n terms, a represents the first term of the progression, r is the common ratio and $r \neq 1$.

Sum to Infinity

If the common ratio lies between -1 and 1 then the geometric progression converges, and it has a sum to infinity. To calculate the sum to infinity, we use the formula below,

$$S_\infty = \frac{a}{1-r}$$

Let's look at some past paper questions.

1. The sum to infinity of a geometric progression is 9 times the sum of the first four terms. Given that the first term is 12 find the value of the fifth term. (9709/12/M/J/19 number 10)

   Let's write out, mathematically, the information we have been given,

   $$S_\infty = 9S_4$$

   We have to use the information above to find u_5,

   $$u_5 = ar^4$$

   We already have a so we need to find r^4.

   Use the formulae for S_n and S_∞ to substitute,

   $$\begin{aligned} S_\infty &= 9S_4 \\ \frac{a}{1-r} &= \frac{9a\left(1-r^4\right)}{1-r} \\ \frac{12}{1-r} &= \frac{9(12)\left(1-r^4\right)}{1-r} \\ \frac{12}{1-r} &= \frac{108\left(1-r^4\right)}{1-r} \end{aligned}$$

   Get rid of the denominator,

   $$12 = 108(1-r^4)$$

   Expand the brackets and simplify,

   $$\begin{aligned} 12 &= 108 - 108r^4 \\ 108r^4 &= 96 \\ r^4 &= \frac{96}{108} \\ r^4 &= \frac{8}{9} \end{aligned}$$

Now that we have r^4 we can find u_5,

$$u_5 = ar^4$$

$$u_5 = 12\left(\frac{8}{9}\right)$$

$$u_5 = \frac{32}{3}$$

Therefore, the final answer is,

$$u_5 = \frac{32}{3}$$

2. The first term of a geometric progression is $\sin^2\theta$, where $0 < \theta < \frac{1}{2}\pi$. The second term of the progression is $\sin^2\theta\cos^2\theta$. Given that the progression is geometric, find the sum to infinity. (9709/13/M/J/20 number 8)

Let's write out the information we have been given,

$$a = \sin^2\theta \qquad u_2 = \sin^2\theta\cos^2\theta$$

The formula for sum to infinity is,

$$S_\infty = \frac{a}{1-r}$$

We already have a, we need to find r,

$$r = \frac{u_2}{u_1}$$

$$r = \frac{\sin^2\theta\cos^2\theta}{\sin^2\theta}$$

$$r = \cos^2\theta$$

Substitute a and r into the formula for sum to infinity,

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{\sin^2\theta}{1-\cos^2\theta}$$

Use the identity $\sin^2\theta \equiv 1 - \cos^2\theta$ to simplify,

$$S_\infty = \frac{\sin^2\theta}{1-\cos^2\theta}$$

$$S_\infty = \frac{\sin^2\theta}{\sin^2\theta}$$

$$S_\infty = 1$$

Therefore, the final answer is,

$$S_\infty = 1$$

3. The first, second and third terms of a geometric progression are $2p+6$, $-2p$ and $p+2$ respectively, where p is positive. Find the sum to infinity of the progression. (9709/12/O/N/20 number 2)

Let's write out the information we have been given,

$$u_1 = 2p+6 \qquad u_2 = -2p \qquad u_3 = p+2$$

The above can be written as,

$$a = 2p+6 \qquad ar = -2p \qquad ar^2 = p+2$$

Note: We have just used the formula for the nth term.

To find the sum to infinity, we need to find a and r by first evaluating p. To evaluate p we can make two equations for r,

$$r = \frac{ar}{a} \qquad r = \frac{ar^2}{ar}$$

$$r = \frac{-2p}{2p+6} \qquad r = \frac{p+2}{-2p}$$

Eqaute the two equations together,

$$\frac{-2p}{2p+6} = \frac{p+2}{-2p}$$

Solve for p,

$$\frac{-2p}{2p+6} = \frac{p+2}{-2p}$$
$$(-2p)(-2p) = (p+2)(2p+6)$$
$$4p^2 = 2p^2 + 10p + 12$$
$$4p^2 - 2p^2 - 10p - 12 = 0$$
$$2p^2 - 10p - 12 = 0$$
$$p^2 - 5p - 6 = 0$$
$$(p+1)(p-6) = 0$$
$$p = -1, p = 6$$
$$p = 6$$

Note: We disregard $p = -1$ because the question states that p is positive.

Substitute p to find r,

$$r = \frac{-2p}{2p+6}$$
$$r = \frac{-2(6)}{2(6)+6}$$
$$r = -\frac{2}{3}$$

Substitute p to find a,

$$a = 2p + 6$$
$$a = 2(6) + 6$$
$$a = 18$$

Now that we have a and r, we can find the sum to infinity,

$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{18}{1-\left(\frac{-2}{3}\right)}$$
$$S_\infty = 10.8$$

Therefore, the final answer is,

$$S_\infty = 10.8$$