GCE A Level Maths 9709

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1.3 Coordinate Geometry

In this topic we will learn how to:

- understand that the equation, $(x-a)^2+(y-b)^2=r^2$ represents the circle with centre (a,b) and radius r
- use the expanded form $x^2+y^2+2gx+2fy+c=0$
- use algebraic methods to solve problems involving lines and circles

Note: Knowledge of geometrical properties of a circle, such as circle theorems, is assumed.

Equation of a Circle

The equation of a circle is written in the form,

$$(x-a)^2+(y-b)^2=r^2$$

Where (a,b) represents the coordinates of the centre of the circle, and r represents the radius.

The equation of a circle can also be written as the expanded form,

$$x^2+y^2+2gx+2fy+c=0$$

Where g, f and c are constants.

To get from the expanded form to the generic form, we complete the square. See example number 1 below.

Let's look at some past paper questions.

1. The equation of the circle with centre C is $x^2 + y^2 - 8x + 4y - 5 = 0$ (9709/12/M/J/20 number 11)

   (a) Find the radius of the circle and coordinates of C.

$$x^2 + y^2 - 8x + 4y - 5 = 0$$

We have been given the expanded equation of the circle. We must complete the square to get to the generic form.

Start by putting terms in x together and terms in y together,

$$x^2 - 8x + y^2 + 4y - 5 = 0$$

Take any constants to the right-hand side,

$$x^2 - 8x + y^2 + 4y = 5$$

Put one set of brackets on the terms in x and one set on the terms in y,

$$(x^2 - 8x) + (y^2 + 4y) = 5$$

Complete the square for both the brackets, separately,

$$(x-4)^2 - 16 + (y+2)^2 - 4 = 5$$

Move all constants to the right-hand side,

$$(x-4)^2 + (y+2)^2 = 5 + 4 + 16$$

Simplify,

$$(x-4)^2 + (y+2)^2 = 25$$

$$(x-4)^2 + (y-(-2))^2 = 25$$

Now that we have the generic equation of the circle, we can read of the coordinates of C,

$$C(4, -2)$$

Let's calculate the radius,

$$r^2 = 25$$

$$\sqrt{r^2} = \sqrt{25}$$

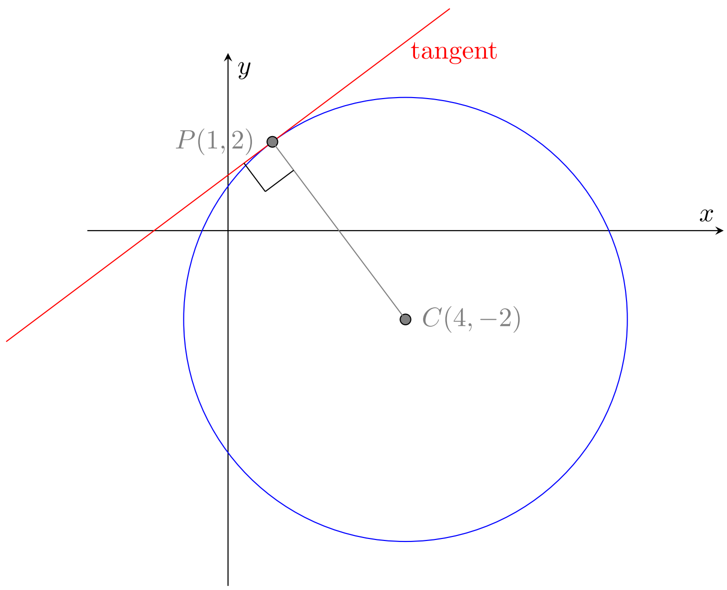
$$r = 5$$

Therefore, the final answer is,

$$C(4, -2) \qquad r = 5$$

(b) The point $P(1, 2)$ lies on the circle. Show that the equation of the tangent to the circle at P is $4y = 3x + 5$.

Let's sketch a diagram of the problem,



To find the equation of the tangent, we need to find its gradient first. The tangent is perpendicular to CP, so let's find the gradient of CP,

$$\text{gradient of CP} = \frac{2+2}{1-4}$$

$$\text{gradient of CP} = -\frac{4}{3}$$

Since CP is perpendicular to the tangent,

$$m_1 \times m_2 = -1$$

$$-\frac{4}{3} \times m_2 = -1$$

$$m_2 = \frac{3}{4}$$

Therefore, the gradient of the tangent is,

$$\frac{3}{4}$$

Now let's find the equation of the tangent, and remember that the tangent passes through P**,**

$$m = \frac{3}{4} \qquad P(1,2)$$

$$y = mx + c$$

$$2 = \frac{3}{4}(1) + c$$

$$2 = \frac{3}{4} + c$$

$$c = 2 - \frac{3}{4}$$

$$c = \frac{5}{4}$$

$$y = \frac{3}{4}x + \frac{5}{4}$$

Multiply through by 4**,**

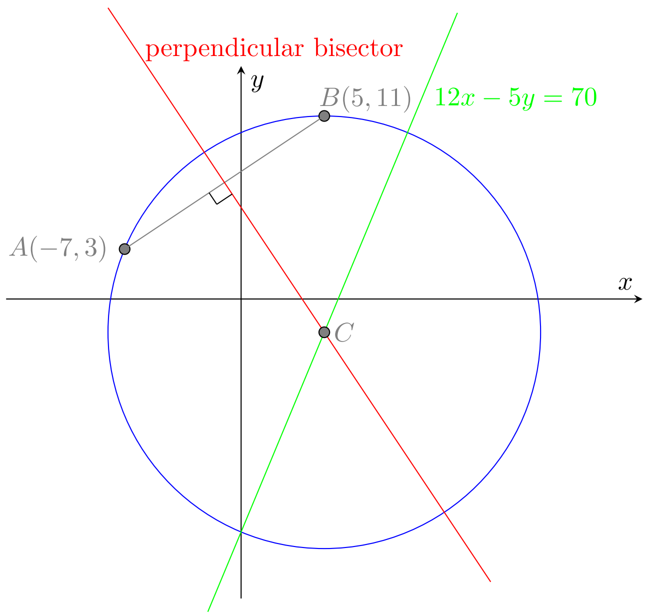
$$4y = 3x + 5$$

Therefore, the final answer is,

$$4y = 3x + 5$$

2. The coordinates of two points A and B are $(-7,3)$ and $(5,11)$ respectively. The perpendicular bisector of AB has the equation $3x+2y=11$. A circle passes through A and B and its center lies on the line $12x-5y=70$. Find an equation of the circle. (9709/13/M/J/20 number 10)

   Let's sketch a diagram of the problem,



The perpendicular bisector and the line $12x - 5y = 70$, intersect at the center of the circle, C. So let's solve simultaneously to find the coordinates of C,

$$3x + 2y = 11 \qquad\qquad 12x - 5y = 70$$

$$2y = 11 - 3x$$
$$y = \frac{11 - 3x}{2}$$

$$12x - 5y = 70$$
$$12x - 5\left(\frac{11 - 3x}{2}\right) = 70$$
$$12x - \frac{55}{2} + \frac{15}{2}x = 70$$
$$\frac{39}{2}x - \frac{55}{2} = 70$$
$$\frac{39}{2}x = 70 + \frac{55}{2}$$
$$\frac{39}{2}x = \frac{195}{2}$$
$$x = 5$$

$$y = \frac{11 - 3x}{2}$$
$$y = \frac{11 - 3(5)}{2}$$
$$y = -2$$

Therefore, the coordinates of the center are,

$$C(5, -2)$$

Now let's find the radius. Since B lies on the circle, the distance from C to B is equal to the radius,

$$B(5, 11) \qquad C(5, -2)$$
$$r = \sqrt{(5-5)^2 + (-2-11)^2}$$
$$r = 13$$

Therefore, the equation of the circle is,

$$(x - 5)^2 + (y - (-2))^2 = 13^2$$
$$(x-5)^2 + (y+2)^2 = 169$$

Therefore, the final answer is,

$$(x-5)^2 + (y+2)^2 = 169$$

3. A circle has center at the point $B(5, 1)$. The point $A(-1, -2)$ lies on the circle. Find the equation of the circle. (9709/12/O/N/20 number 9)

   We already have the center, so we need to find the radius. The distance from the center, B, to the point A is equal to the radius,

$$B(5, 1) \qquad A(-1, -2)$$
$$r = \sqrt{(5+2)^2 + (1+2)^2}$$
$$r = 3\sqrt{5}$$

   Therefore, the equation of the circle is,

$$(x-5)^2 + (y-1)^2 = (3\sqrt{5})^2$$
$$(x-5)^2 + (y-1)^2 = 45$$

Therefore, the final answer is,

$$(x-5)^2 + (y-1)^2 = 45$$

4. A circle with center C has equation $(x-8)^2+(y-4)^2 = 100$. (9709/13/O/N/20 number 11)

   (a) Show that the point $T(-6, 6)$ is outside the circle.

   $$(x-8)^2 + (y-4)^2 = 100$$

   Substitute the coordinates of point T into the equation of the circle,

   $$(x-8)^2 + (y-4)^2$$
   $$(-6-8)^2 + (6-4)^2$$
   $$104$$
   $$r^2 = 104$$
   $$r = 2\sqrt{26}$$
   $$r = 10.2$$

   The radius of our circle is $\sqrt{100}$, which is 10, since the line from the center to T is larger than our radius T lies outside the circle,

   $$10.2 > 10$$

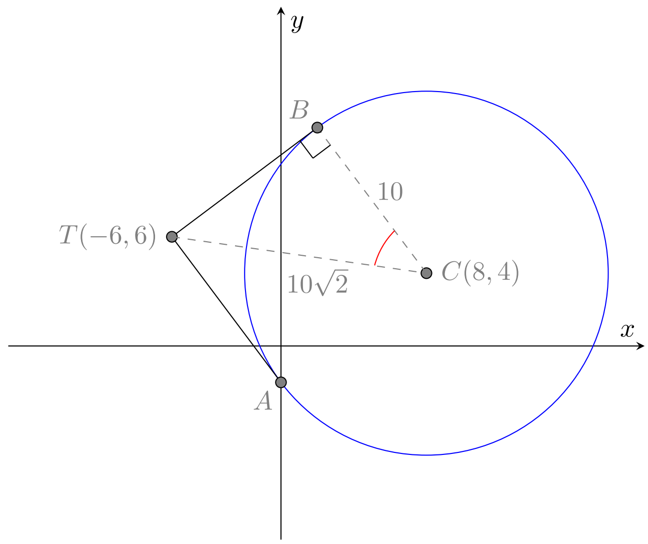
   Therefore, the final answer is,

   $$10.2 > 10 \text{ hence } T(-6, 6) \text{ lies outside the circle}$$

Two tangents from T to the circle are drawn.

(b) Show that the angle between one of the tangents and CT is exactly $45°$.

Let's sketch a diagram of the problem,



Let's start by finding the length of CT,

$$C(8,4) \qquad\qquad T(-6,6)$$

$$CT = \sqrt{(8+6)^2 + (4-6)^2}$$

$$CT = 10\sqrt{2}$$

Let's use Pythagoras to find the angle θ,

$$\cos\theta = \frac{10}{10\sqrt{2}}$$

$$\theta = \cos^{-1}\left(\frac{10}{10\sqrt{2}}\right)$$

$$\theta = 45°$$

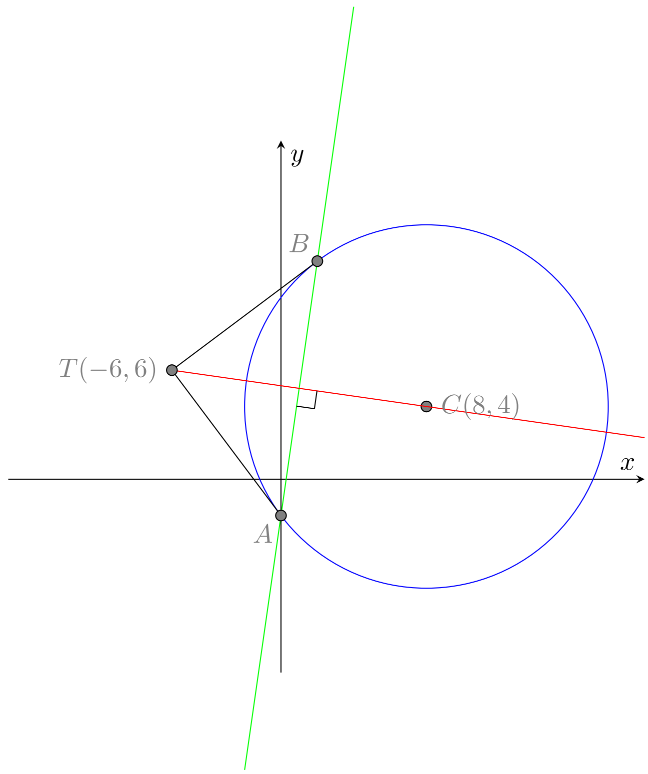
Therefore, the final answer is,

$$\theta = 45°$$

The two tangents touch the circle at A and B.

(c) Find the equation of the line AB, giving your answer in the form $y = mx + c$.

Let's sketch a diagram of the problem,



Line AB **is perpendicular to** CT**, so let's find the gradient of** CT**,**

$$C(8,4) \qquad T(-6,6)$$

$$\text{gradient of CT } = \frac{6-4}{-6-8}$$

$$\text{gradient of CT } = -\frac{1}{7}$$

Since AB **is perpendicular to** CT**,**

$$m_1 \times m_2 = -1$$

$$-\frac{1}{7} \times m_2 = -1$$

$$m_2 = 7$$

Therefore, the gradient of AB **is,**

$$7$$

The midpoint of CT lies on the line AB, so let's find M,

$$M = \left(\frac{8-6}{2}, \frac{4+6}{2}\right)$$

$$M = (1, 5)$$

Therefore, the line AB passes through,

$$M(1, 5)$$

Now let's find the equation of the line AB,

$$m = 7 \qquad\qquad M(1, 5)$$

$$y = mx + c$$

$$5 = 7(1) + c$$

$$5 = 7 + c$$

$$c = 5 - 7$$

$$c = -2$$

$$y = 7x - 2$$

Therefore, the final answer is,

$$y = 7x - 2$$

(d) Find the x-coordinates of A and B.

A and B are the points of intersection of the line AB and the circle. Therefore, we will solve simultaneously,

$$y = 7x - 2 \qquad\qquad (x-8)^2 + (y-4)^2 = 100$$

Substitute $y = 7x - 2$ into the equation of the circle,

$$(x-8)^2 + (y-4)^2 = 100$$

$$(x-8)^2 + (7x-2-4)^2 = 100$$

$$(x-8)^2 + (7x-6)^2 = 100$$

Expand the quadratics,

$$x^2 - 16x + 64 + 49x^2 - 84x + 36 = 100$$

Put all the terms on one side,

$$x^2 + 49x^2 - 16x - 84x + 64 + 36 - 100 = 0$$

Solve the quadratic equation,

$$50x^2 - 100x = 0$$

$$50x(x - 2) = 0$$

$$x = 0 \qquad x = 2$$

Therefore, the x-coordinates of A and B are,

$$x = 0 \qquad x = 2$$