GCE A Level Maths 9709

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April 2023

1.7 Differentiation

In this topic we will learn how to:

- apply differentiation to increasing and decreasing functions

Increasing and Decreasing Functions

An increasing function is one whose gradient is always positive. In other words $\frac{dy}{dx} > 0$. A decreasing function is one whose gradient is always negative. In other words $\frac{dy}{dx} < 0$.

To determine whether a function is an increasing or decreasing function, we need to find the gradient function first. So we differentiate the original equation. Based on the gradient function we can determine whether $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$. Once we know that, we can decide whether it is an increasing or decreasing function.

Let's look at some past paper questions on increasing and decreasing functions.

1. The function f is defined by $f(x) = \frac{1}{3x+2} + x^2$ for $x < -1$. Determine whether f is an increasing function, a decreasing function or neither. (9709/12/F/M/20 number 10)

$$f(x) = \frac{1}{3x+2} + x^2$$

Bring the denominator up to the numerator in the fraction, using law of indices,

$$f(x) = (3x+2)^{-1} + x^2$$

Differentiate $f(x)$,

$$f'(x) = -3(3x+2)^{-2} + 2x$$

Note: If you're not able to use the shortcut of the chain rule that allows you to differentiate by sight, it is still fine to use the chain rule.

Rewrite the differential but with positive indices,

$$f'(x) = \frac{-3}{(3x+2)^2} + 2x$$

If you look at the first term, you will notice that for all values of x, the denominator is will always be positive, since it is squared. So the first term will always be negative (since numerator is negative). The second term will always be negative since the domain of the function is $x < -1$. This means that our gradient will always be negative for $x < -1$,

$$f'(x) < 0$$

Therefore, the final answer is,

$$f(x) \textbf{ is a decreasing function.}$$

2. The function f is defined by $f(x) = x^5 - 10x^3 + 50x$ for $x \in \mathbb{R}$. Given that $5y^2 - 30y + 50$ can be expressed in the form $5(y-3)^2 + 5$, determine whether f is an increasing function, a decreasing function, or neither. (9709/13/O/N/21 number 3)

$$f(x) = x^5 - 10x^3 + 50x$$

Let's start by differentiating f,

$$f'(x) = 5x^4 - 30x^3 + 50$$

Let's compare the given equation to our differential,

$$5y^2 - 30y + 50$$

$$5x^4 - 30x^3 + 50$$

You will notice that our differential is very similar to the given equation,

$$5{\color{blue}y}^2 - 30{\color{blue}y} + 50$$

$$5\left({\color{blue}x^2}\right)^2 - 30x\left({\color{blue}x^2}\right) + 50$$

Therefore, we can rewrite our differential as,

$$5(x^2)^2 - 30x\left(x^2\right) + 50$$

$$5(x^2 - 3)^2 + 5$$

Therefore, our differential is,

$$f'(x) = 5(x^2 - 3)^2 + 5$$

If you analyse the differential, you will notice that, the first term is squared so it will always be positive for all values of x and the second term is positive. Therefore, our gradient function will always be positive,

$$f'(x) > 0$$

Therefore, the final answer is,

$$f(x) \textbf{ is an increasing function.}$$

3. The function f is defined by $f(x) = \frac{1}{3}(2x-1)^{\frac{3}{2}} - 2x$ for $\frac{1}{2} < x < a$. It is given that f is a decreasing function. Find the maximum possible value of the constant a. (9709/13/M/J/21 number 2)

$$f(x) = \frac{1}{3}(2x-1)^{\frac{3}{2}} - 2x$$

We are given that f is a decreasing function, therefore,

$$f'(x) < 0$$

Let's find the differential of f,

$$f'(x) = \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)(2)\left(2x-1\right)^{\frac{1}{2}} - 2$$

$$f'(x) = (2x-1)^{\frac{1}{2}} - 2$$

Let's use the idea that $f'(x) < 0$,

$$(2x-1)^{\frac{1}{2}} - 2 < 0$$

Solve for x,

$$(2x-1)^{\frac{1}{2}} - 2 < 0$$

$$(2x-1)^{\frac{1}{2}} < 2$$

Square both sides,

$$2x - 1 < 4$$

Make x the subject of the formula,

$$2x < 5$$

$$x < \frac{5}{2}$$

Therefore, the maximum possible value of a is,

$$a = \frac{5}{2}$$