# GCE A Level Maths 9709

SMIYL

April 2023

## 1.7 Differentiation

In this topic we will learn how to:

- apply differentiation to rates of change

### Rates of Change

**Rate of change refers to the speed at which a variable changes. We can use differentiation to determine the rate at which a variable changes. If we were to investigate the rate at which a variable $x$ was changing we would denote this using the notation,**

$$\frac{dx}{dt}$$

**To find the rate at which a variable is changing we will use the chain rule. Let's walk through some past paper questions.**

1. A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^3}$ and $A(1, -3)$ lies on the curve. A point is moving along the curve and at $A$ the $y$-coordinate of the point is increasing at 3 units per second. Find the rate of increase at $A$ of the $x$-coordinate of the point. (9709/12/F/M/21 number 6)

   **Let's write out, mathematically, all the information we have been given,**

   $$\frac{dy}{dx} = \frac{6}{(3x-2)^3} \qquad \frac{dy}{dt} = 3 \qquad \frac{dx}{dt} = ? \qquad A(1, -3)$$

   **The question requires us to find $\frac{dx}{dt}$. Let's define a chain rule in terms of $\frac{dx}{dt}$,**

   $$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

**We already have $\frac{dy}{dt}$. Let's find $\frac{dx}{dy}$, using $\frac{dy}{dx}$,**

$$\frac{dy}{dx} = \frac{6}{(3x-2)^3}$$

$$\frac{dx}{dy} = \frac{(3x-2)^3}{6}$$

**Note: Rates of change behave like fractions. So we can flip them to find their reciprocal.**

**Let's substitute the $x$-value at $A(1,-3)$ into $\frac{dx}{dy}$,**

$$\frac{dx}{dy} = \frac{(3x-2)^3}{6}$$

$$\frac{dx}{dy} = \frac{(3(1)-2)^3}{6}$$

$$\frac{dx}{dy} = \frac{1}{6}$$

**Now let's go back to our chain rule,**

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

**Evaluate $\frac{dx}{dt}$,**

$$\frac{dx}{dt} = \frac{1}{6} \times 3$$

$$\frac{dx}{dt} = \frac{1}{2}$$

**Therefore, the final answer is,**

$$\frac{dx}{dt} = \frac{1}{2} \textbf{ units per second}$$

2. A weather balloon in the shape of a sphere is being inflated by a pump. The volume of the balloon is increasing at a constant rate of 600 $\text{cm}^3$ per second. The balloon was empty at the start of pumping. (9709/12/M/J/20 number 3)

   (a) Find the radius of the balloon after 30 seconds.

**To find the radius we need the formula for volume of a sphere,**

$$V = \frac{4}{3}\pi r^3$$

**Note: You do not need to know it, it is given the list of formulae booklet MF19.**

**We are told that the volume is increasing at a constant rate of** $600$ **cm**$^3$ **per second. So we can multiply by that number by** $30$ **to determine the volume after** $30$ **seconds,**

$$V = 600 \times 30$$

$$V = 18\ 000$$

**Now let's equate** $18\ 000$ **to the formula for volume of a sphere and evaluate** $r$**,**

$$V = 18\ 000 \qquad\qquad V = \frac{4}{3}\pi r^3$$

$$18\ 000 = \frac{4}{3}\pi r^3$$

**Make** $r^3$ **the subject of the formula,**

$$r^3 = \frac{3 \times 18\ 000}{4\pi}$$

$$r^3 = \frac{13\ 500}{\pi}$$

**Solve for** $r$**,**

$$r = \left(\frac{13\ 500}{\pi}\right)^{\frac{1}{3}}$$

$$r = 16.2577821$$

$$r = 16.3$$

**Therefore, the final answer is,**

$$r = 16.3 \textbf{ cm}$$

(b) Find the rate of increase of the radius after 30 seconds.

**The question requires us to find,**

$$\frac{dr}{dt}$$

**Let's write the information from the stem of the question that can help us to define a chain rule,**

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = 600, \quad \textbf{after 30 seconds } r = 16.2578$$

**Let's construct a chain for** $\frac{dr}{dt}$**,**

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

**We already have** $\frac{dV}{dt}$**. To find** $\frac{dr}{dV}$ **let's first find** $\frac{dV}{dr}$**,**

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = (3)\left(\frac{4}{3}\right)\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2$$

**Flip** $\frac{dV}{dr}$ **to get** $\frac{dr}{dV}$**,**

$$\frac{dr}{dV} = \frac{1}{4\pi r^2}$$

**Substitute** $r = 16.2578$**,**

$$\frac{dr}{dV} = \frac{1}{4\pi(16.2578)^2}$$

$$\frac{dr}{dV} = \frac{1}{4\pi(16.2578)^2}$$

$$\frac{dr}{dV} = 0.00030106937$$

**Note: Use the either the exact value of** $r$ **or the value of** $r$ **correct to four decimal places or more to get the correct answer.**

**Let's go back to the chain rule we defined,**

$$\frac{dr}{dt} = {\color{blue}\frac{dr}{dV}} \times {\color{green}\frac{dV}{dt}}$$

**Substitute and evaluate $\frac{dr}{dt}$,**

$$\frac{dr}{dt} = 0.00030106937 \times 600$$

$$\frac{dr}{dt} = 0.181$$

**Therefore, the final answer is,**

$$\frac{dr}{dt} = 0.181 \textbf{ cm per second}$$

3. The Volume $V$ $m^3$ of a large circular mound of iron ore of radius $r$ m is modelled by the equation $V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1$ for $r \geq 2$. Iron ore is added to the mound at a constant rate of 1.5 $m^3$ per second.

   (a) Find the rate at which the radius of the mound is increasing at the instant when the radius is 5.5 m. (9709/12/O/N/21 number 9)

   **Let's write out, mathematically, all the information we have been given,**

$$V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1 \qquad \frac{dV}{dt} = 1.5 \qquad \frac{dr}{dt} = ? \qquad r = 5.5$$

   **Using the information above, let's define a chain rule for $\frac{dr}{dt}$**

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

   **We already have $\frac{dV}{dt}$. To find $\frac{dr}{dV}$ let's first find $\frac{dV}{dr}$,**

$$V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1$$

$$\frac{dV}{dr} = (3)\left(\frac{3}{2}\right)\left(r - \frac{1}{2}\right)^2$$

$$\frac{dV}{dr} = \frac{9}{2}\left(r - \frac{1}{2}\right)^2$$

   **Flip $\frac{dV}{dr}$ to get $\frac{dr}{dV}$,**

$$\frac{dr}{dV} = \frac{2}{9\left(r - \frac{1}{2}\right)^2}$$

**Substitute $r = 5.5$,**

$$\frac{dr}{dV} = \frac{2}{9\left(5.5 - \frac{1}{2}\right)^2}$$
$$\frac{dr}{dV} = \frac{2}{225}$$

**Let's go back to the chain rule we defined,**

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

**Substitute and evaluate $\frac{dr}{dt}$,**

$$\frac{dr}{dt} = \frac{2}{225} \times 1.5$$
$$\frac{dr}{dt} = \frac{1}{75}$$

**Therefore, the final answer is,**

$$\frac{dr}{dt} = \frac{1}{75} \textbf{ m per second}$$

(b) Find the volume of the mound at the instant when the radius is increasing at 0.1 m per second.

**Let's write out, mathematically, all the information we have been given,**

$$V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1 \qquad \frac{dV}{dt} = 1.5 \qquad \frac{dr}{dt} = 0.1 \qquad r = 5.5$$

**Using the information above, let's define a chain rule for $\frac{dr}{dt}$,**

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

**Substitute and evaluate $\frac{dr}{dV}$,**

$$0.1 = \frac{dr}{dV} \times 1.5$$
$$\frac{dr}{dV} = \frac{0.1}{1.5}$$
$$\frac{dr}{dV} = \frac{1}{15}$$

**Remember that in part $(a)$ we found that,**

$$\frac{dr}{dV} = \frac{2}{9\left(r - \frac{1}{2}\right)^2}$$

**Let's equate $\frac{dr}{dV}$ to $\frac{1}{15}$,**

$$\frac{1}{15} = \frac{2}{9\left(r - \frac{1}{2}\right)^2}$$

**Cross multiply and simplify,**

$$\frac{2(15)}{9} = \left(r - \frac{1}{2}\right)$$

$$\frac{10}{3} = \left(r - \frac{1}{2}\right)^2$$

**Take the square roots of both sides,**

$$\pm\sqrt{\frac{10}{3}} = \sqrt{\left(r - \frac{1}{2}\right)^2}$$

$$\pm\sqrt{\frac{10}{3}} = r - \frac{1}{2}$$

**Make $r$ the subject of the formula,**

$$r = \frac{1}{2} \pm \sqrt{\frac{10}{3}}$$

$$r = \frac{1}{2} + \sqrt{\frac{10}{3}}$$

**Note: We disregard $r = \frac{1}{2} - \sqrt{\frac{10}{3}}$ because it gives a negative radius. Radius is a measurement, therefore, it is a positive quantity.**

**We can now substitute the radius into the formula for volume, to find the volume of the mound,**

$$V = \frac{3}{2}\left(r - \frac{1}{2}\right)^3 - 1$$

$$V = \frac{3}{2}\left(\frac{1}{2} + \sqrt{\frac{10}{3}} - \frac{1}{2}\right)^3 - 1$$

$$V = \frac{-3 + 5\sqrt{30}}{3}$$

$$V = 8.13$$

**Therefore, the final answer is,**

$$V = 8.13 \ \mathbf{m}^3$$