GCE A Level Maths 9709

SMIYL

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1.6 Series

In this topic we will learn how to:

- recognise an arithmetic progression
- use the formulae for nth term and for the sum of the first n terms to solve problems involving arithmetic progressions

Arithmetic Progressions

An arithmetic progression is a sequence of numbers in which each differs from the preceding one by a common difference. To find the nth **term under an arithmetic progression, we use the formula,**

$$u_n = a + (n-1)d$$

Where u_n **represents the** nth **term,** a **represents the first term of the progression,** n **represents the position of the** nth **term and** d **represents the common difference.**

To find the sum of the first n **terms under an arithmetic progression, we use the following two formulae,**

$$S_n = \frac{1}{2}n(a+l)$$

$$S_n = \frac{1}{2}n(2a+(n-1)d)$$

Where S_n **represents sum of first** n **terms,** a **represents the first term of the progression,** l **represents the last term of the of the progression,** d **is the common difference.**

Note: Only use the first formula if you know the last term of the progression, otherwise use the second formula.

Let's look at some past paper questions.

1. The sum of the first nine terms of an arithmetic progression is 117. The sum of the next four terms is 91. Find the first term and the common difference of the progression. (9709/11/M/J/20 number 1)

   Let's write out, mathematically, the information we have been given,

$$S_9 = 117 \qquad S_{13} = S_9 + 91$$

   We can make two equations, in terms of a and d that we can solve simultaneously,

$$S_9 = 117 \qquad S_{13} = S_9 + 91$$

$$S_9 = 117 \qquad S_{13} = 117 + 91$$

$$S_9 = 117 \qquad S_{13} = 208$$

   Use the formula for sum of first n terms, to create two equations,

$$\frac{1}{2}(9)(2a + (9-1)d) = 117 \qquad \frac{1}{2}(13)(2a + (13-1)d) = 208$$

   Solve the two equations simultaneously,

$$\frac{9}{2}(2a + 8d) = 117 \qquad \frac{13}{2}(2a + 12d) = 208$$

$$9a + 36d = 117 \qquad 13a + 78d = 208$$

$$9a = 117 - 36d$$

$$a = \frac{117 - 36d}{9}$$

$$13a + 78d = 208$$

$$13\left(\frac{117 - 36d}{9}\right) + 78d = 208$$

$$169 - 52d + 78d = 208$$

$$169 + 26d = 208$$

$$26d = 208 - 169$$

$$26d = 39$$

$$d = \frac{3}{2}$$

$$a = \frac{117 - 36\left(\frac{3}{2}\right)}{9}$$
$$a = 7$$

Therefore, the final answer is,

$$a = 7 \qquad d = \frac{3}{2}$$

2. The nth term of an arithmetic progression is $\frac{1}{2}(3n - 15)$. Find the value of n for which the sum of the first n terms is 84. (9709/12/M/J/20 number 4)

   We have been given that,

   $$S_n = 84$$

   Use the formula for S_n,

   $$\frac{1}{2}(n)(2a + (n-1)d) = 84$$

   Let's use the formula for the nth term that we have been given to find a and d,

   $$u_n = \frac{1}{2}(3n - 15)$$

   u_1 **represents the first term. So let's substitute $n = 1$ to find the first term,**

   $$u_1 = \frac{1}{2}(3(1) - 15)$$
   $$u_1 = -6$$
   $$a = -6$$

   Let's find the second term, so we can calculate d,

   $$u_2 = \frac{1}{2}(3(2) - 15)$$
   $$u_2 = -\frac{9}{2}$$

Now let's find d,

$$d = u_2 - u_1$$
$$d = -\frac{9}{2} - (-6)$$
$$d = \frac{3}{2}$$

Now that we have a and d let's solve for n,

$$\frac{1}{2}(n)(2a + (n-1)d) = 84$$
$$\frac{1}{2}(n)(2(-6) + (n-1)(\frac{3}{2}) = 84$$
$$\frac{1}{2}(n)(-12 + \frac{3}{2}n - \frac{3}{2}) = 84$$
$$\frac{1}{2}(n)(\frac{3}{2}n - \frac{27}{2}) = 84$$
$$\frac{3}{4}n^2 - \frac{27}{4}n = 84$$
$$\frac{3}{4}n^2 - \frac{27}{4}n - 84 = 0$$
$$3n^2 - 27n - 336 = 0$$
$$n^2 - 9n - 112 = 0$$
$$(n+7)(n-16) = 0$$
$$n = -7, n = 16$$

Note: Disregard $n = -7$, because n represents positive integers, since it the sum of first n terms.

Therefore, our final answer is,

$$n = 16$$

3. The first term of a progression is $\sin^2\theta$, where $0 < \theta < \frac{1}{2}\pi$. The second term of the progression $\sin^2\theta\cos^2\theta$. It is given that the progression is arithmetic. (9709/13/M/J/20 number 8)

   (a) Find the common difference of the progression in terms of $\sin\theta$.

   Let's write out the information we have been given,

$$a = \sin^2\theta \qquad u_2 = \sin^2\theta\cos^2\theta$$

We can find d using a and u_2,

$$d = u_2 - a$$

$$d = \sin^2\theta\cos^2\theta - \sin^2\theta$$

Replace $\cos^2\theta$ with $1 - \sin^2$,

$$d = \sin^2\theta(1 - \sin^2\theta) - \sin^2\theta$$

$$d = \sin^2\theta - \sin^4\theta - \sin^2\theta$$

$$d = -\sin^4\theta$$

Therefore, the final answer is,

$$d = -\sin^4\theta$$

(b) Find the sum of the first 16 terms when $\theta = \frac{1}{3}\pi$.

$$a = \sin^2\theta \qquad d = -\sin^4\theta$$

Substitute θ with $\frac{1}{3}\pi$ in a and d,

$$a = \sin^2\left(\frac{1}{3}\pi\right) \qquad d = -\sin^4\left(\frac{1}{3}\pi\right)$$

$$a = \frac{3}{4} \qquad d = -\frac{9}{16}$$

Now let's find the sum of the first 16 terms,

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

$$S_{16} = \frac{1}{2}(16)\left(2\left(\frac{3}{4}\right) + (16-1) \times \frac{-9}{16}\right)$$

$$S_{16} = -\frac{111}{2}$$

Therefore, the final answer is,

$$S_{16} = -\frac{111}{2}$$