GCE A Level Maths 9709

SMIYL

April 2023

1.5 Trigonometry

In this topic we will learn how to:

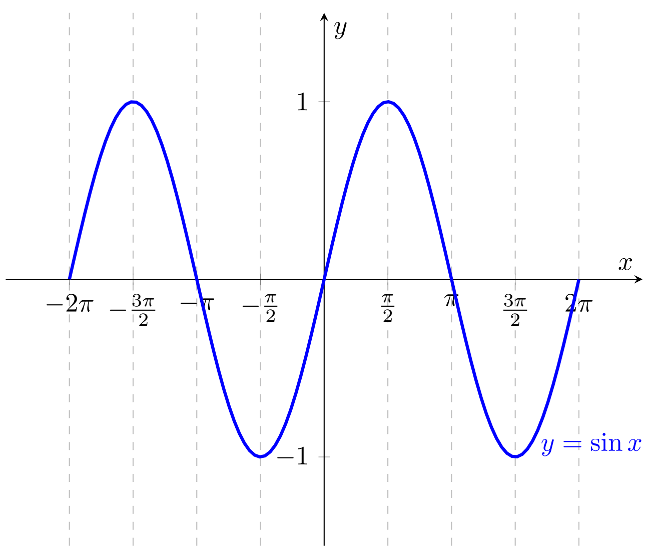
- sketch and use the graphs of sine, cosine and tangent functions
- understand and use the transformations of the graphs of $y = f(x)$ given by $y = f(x) + a$, $y = f(x + a)$, $y = af(x)$ and $y = f(ax)$ and simple combinations of these for trigonometric functions

Trigonometric Functions and their Transformations

Sine Function

The sine function is typically denoted as,

$$y = \sin x$$

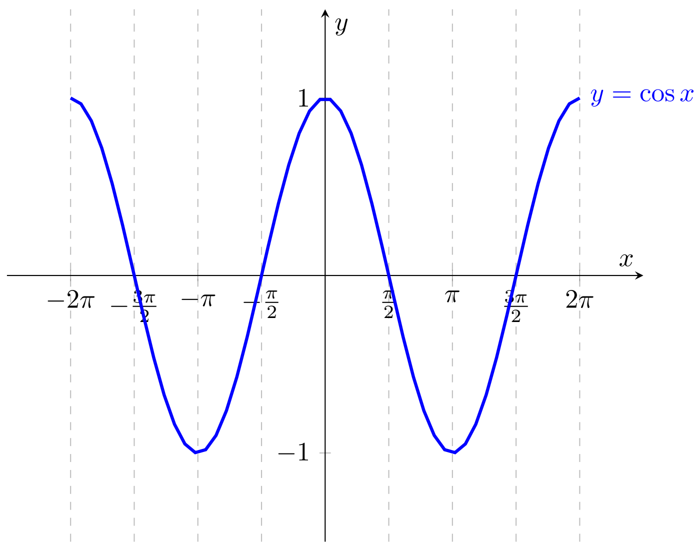


The graph of $y = \sin x$. It has a period of 2π radians i.e it repeats every 2π radians.

Cosine Function

The cosine function is typically denoted as,

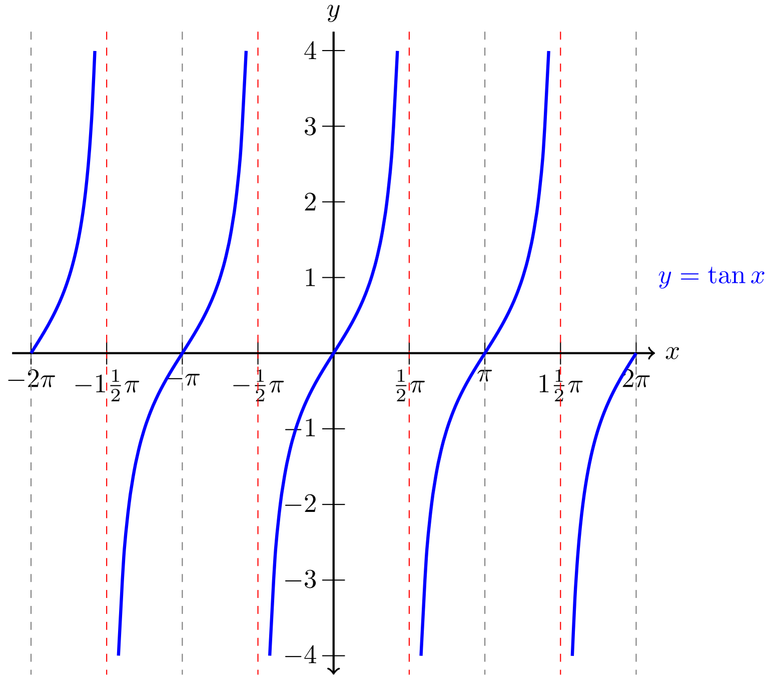
$y = \cos x$



The graph of $y = \cos x$. It has a period of 2π radians i.e it repeats every 2π radians.

The tangent function is typically denoted as,

$$y = \tan x$$



The graph of $y = \tan x$. It has a period of π radians.

Note: The red lines are asymptotes. An asymptote is a line that the function approaches but never reaches.

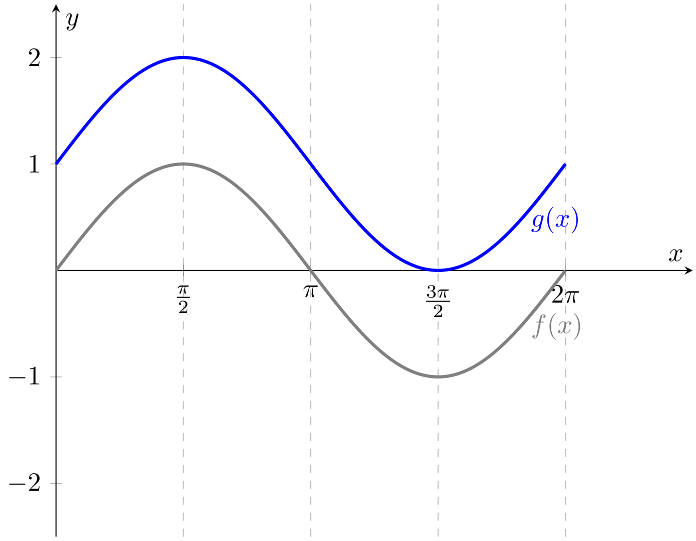
Transformations

$$y = \sin x + a$$

This is a translation in the y direction by a units. Let's look at an example of this transformation.

Example 1

Given that $f(x) = \sin x$ for $0 < x < 2\pi$ sketch the graph of $g(x) = f(x) + 1$ for $0 < x < 2\pi$.



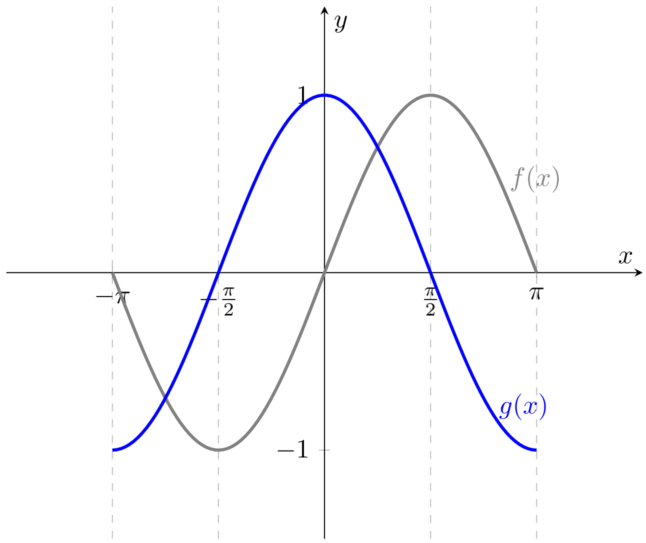
$g(x)$ is a translation by 1 unit in the y direction from $f(x)$

$$y = \sin(x + a)$$

This is a translation in the x direction by $-a$ units. Let's look at an example of this transformation.

Example 2

Given that $f(x) = \sin x$ for $-\pi < x < \pi$ sketch the graph of $g(x) = f\left(x + \frac{\pi}{2}\right)$ for $-\pi < x < \pi$.



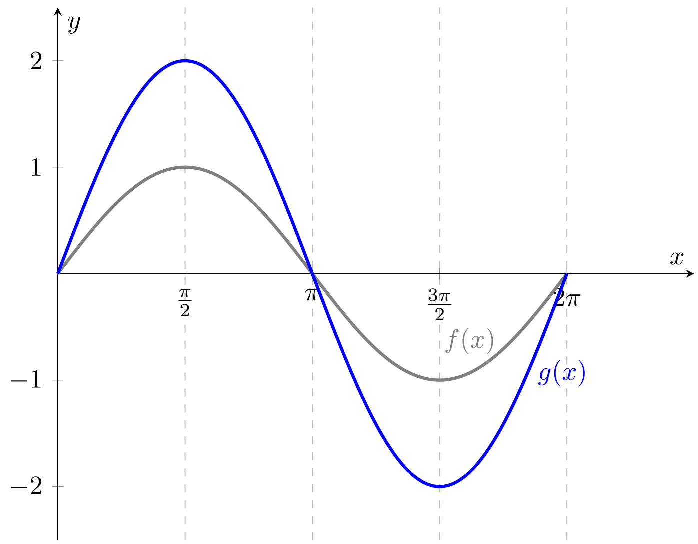
$$g(x) \text{ is a translation by } -\frac{\pi}{2} \text{ units in the } x \text{ direction from } f(x)$$

$$y = a \sin x$$

This is a stretch in the y direction by a stretch factor of a. Let's look at an example of this transformation.

Example 3

Given that $f(x) = \sin x$ for $0 < x < 2\pi$ sketch the graph of $g(x) = 2f(x)$ for $0 < x < 2\pi$.



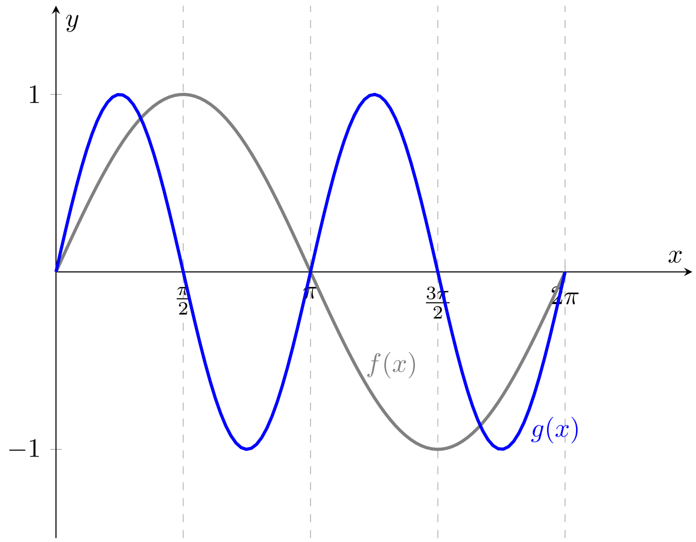
$g(x)$ is a stretch in the y direction by a stretch factor of 2 from $f(x)$.

$$y = \sin(ax)$$

This is a stretch in the x direction by a stretch factor of $\frac{1}{a}$. It represents the number of periods of the graph. Let's look at an example of this transformation.

Example 4

Given that $f(x) = \sin x$ for $0 < x < 2\pi$ sketch the graph of $g(x) = f(2x)$ for $0 < x < 2\pi$.



$g(x)$ is a stretch in the x direction by a stretch factor of $\frac{1}{2}$ from $f(x)$.

Note: Notice how there are two periods of $g(x)$ in the interval $0 < x < 2\pi$.

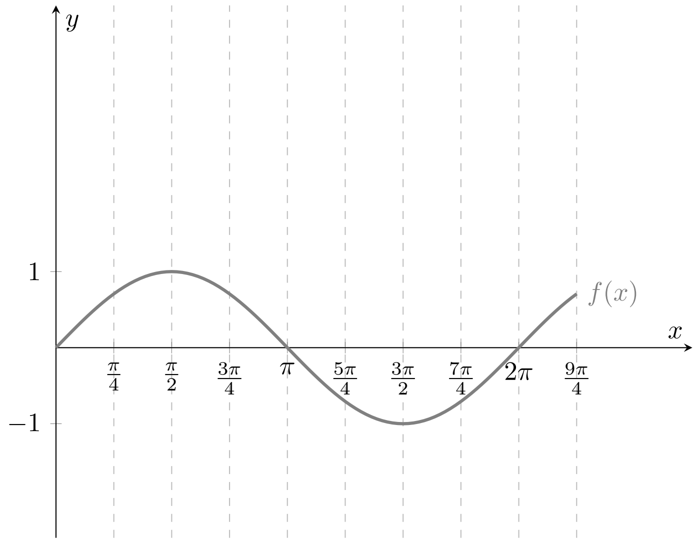
$$y = a\sin(bx + c) + d$$

The above is known as a combined transformation. a represents a stretch in the y direction. b represents a stretch in the x direction. c represents a translation in the x direction. d represents a translation in the y direction. Let's look at an example of a combined transformation.

Example 5

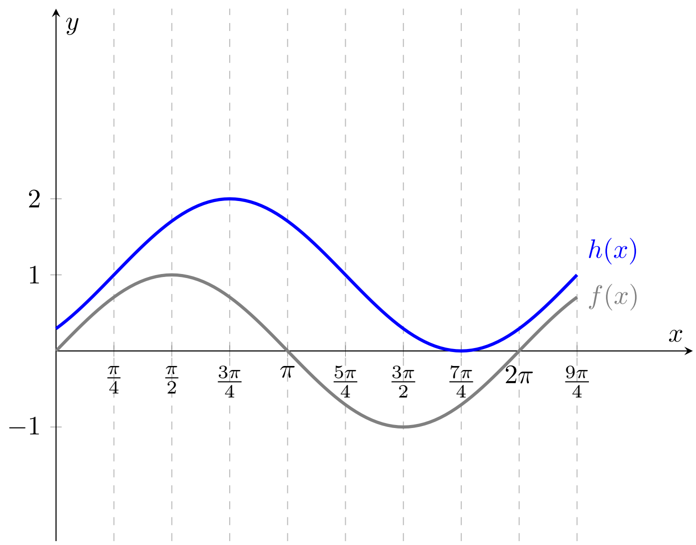
Given that $f(x) = \sin x$ for $0 < x < \frac{5}{2}\pi$ sketch the graph of $g(x) = 3f\left(2x - \frac{\pi}{4}\right) + 1$ for $0 < x < \frac{9}{4}\pi$.

Start by sketching the graph of $f(x)$,



$$y = f(x)$$
$$g(x) = 3f\left(2x - \frac{\pi}{4}\right) + 1$$

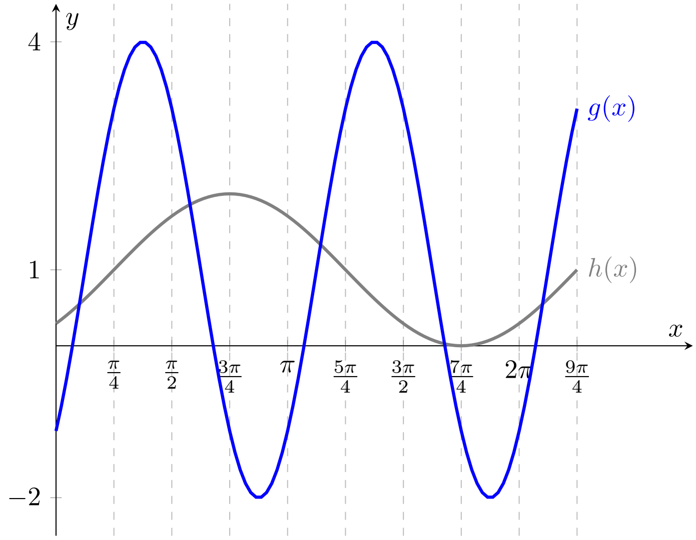
From $f(x)$ to $g(x)$ there is a translation in the x direction by $\frac{\pi}{4}$, then there is a translation in the y direction by 1 unit,



Translation by $\frac{\pi}{4}$ units in the x direction and 1 unit in the y direction from $f(x)$

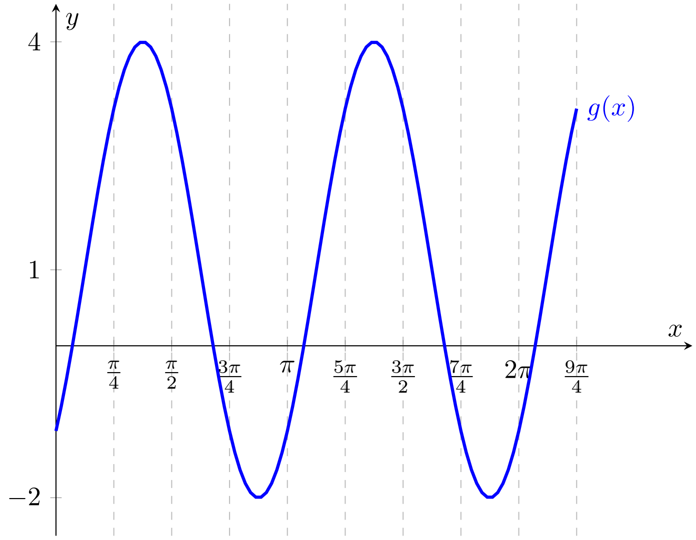
$$g(x) = 3f\left(2x - \frac{\pi}{4}\right) + 1$$

Finally there is a stretch in the x direction by a stretch factor of $\frac{1}{2}$, then a stretch in the y direction by a stretch factor of 3,



A stretch in the x direction by a stretch factor of $\frac{1}{2}$ and a stretch in the x direction by a stretc

Therefore, the graph of $g(x)$,



$$y = g(x)$$

Note: The transformations for $y = \cos x$ and $y = \tan x$ work exactly the same as those for $y = \sin x$.

Finding the range of a trig function

To find the range of a trig function, we have to find the minimum and maximum points. To find the minimum point of a trig function substitute the part in the equation containing the trig function with -1. For example, $y = 2\sin x + 3$.

Replace $\sin x$ with -1,

$$y = 2(-1) + 3$$

$$y = 1$$

Therefore, the minimum point of $y = 2\sin x + 3$ is 1.

To find the maximum point, substitute the part in the equation containing the trig function with 1. For example $y = 3\sin 2x - 1$.

Replace $\sin 2x$ with 1,

$$y = 3(1) - 1$$

$$y = 2$$

Therefore, the maximum point of $y = 3\sin 2x - 1$ is 2.

Note: If the trig function is negative (see past paper question number 1), then replacing the trig function with -1 will give you the maximum point and 1 will give you the minimum point.

Let's look at some past paper questions.

1. The function f is defined by $f(x) = 2 - 3\cos x$ for $0 \leq x \leq 2\pi$. (9709/11/M/J/19 number 9)

   (a) State the range of f.

   $$f(x) = 2 - 3\cos x$$

   To get the range of f, we have to find both the maximum and minimum point of f. To find the maximum point, replace $\cos x$ with -1,

   $$f(x) = 2 - 3(-1)$$

   $$f(x) = 5$$

Therefore, the maximum point of f is,

$$5$$

To find the minimum point, replace $\cos x$ with 1,

$$f(x) = 2 - 3(1)$$

$$f(x) = -1$$

Therefore, the minimum point of f is,

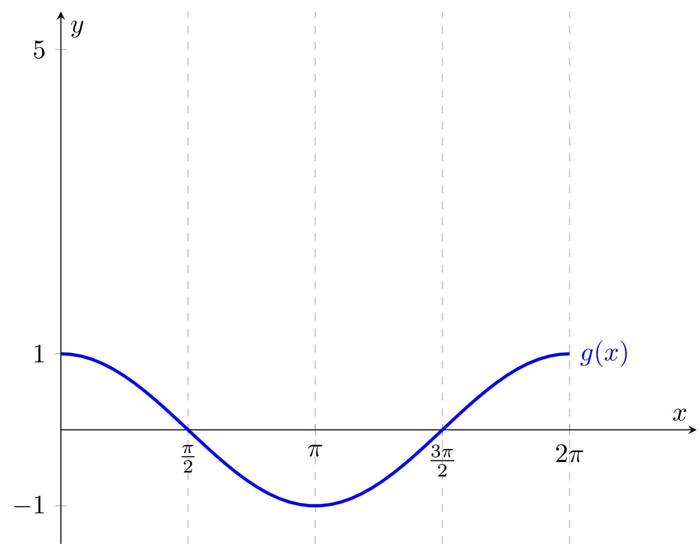
$$-1$$

Therefore, the range of f is,

$$-1 \leq f(x) \leq 5$$

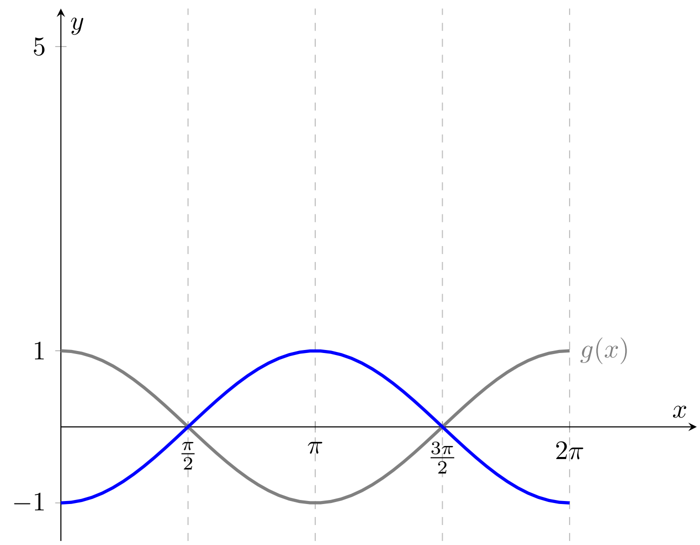
(b) Sketch the graph of $y = f(x)$.

Start by sketching the graph of $y = \cos x$ for $0 \leq x \leq 2\pi$. Let's call this function $g(x)$,



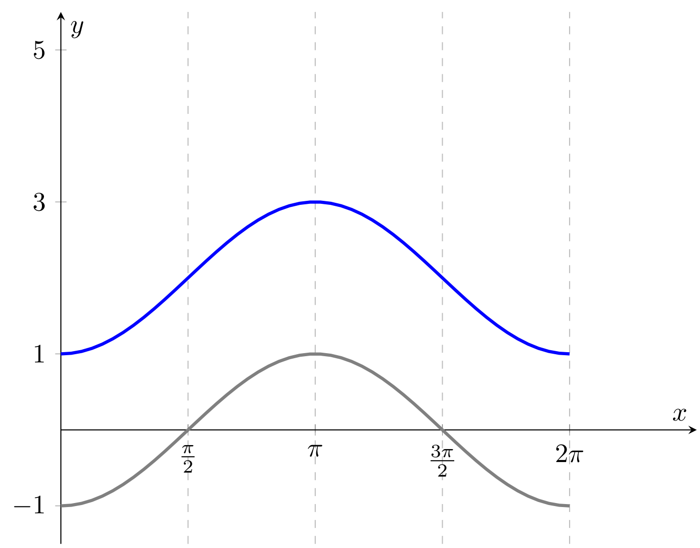
$g(x) = \cos x$

The negative sign on $3\cos x$ means the function has been reflected in the x-axis,



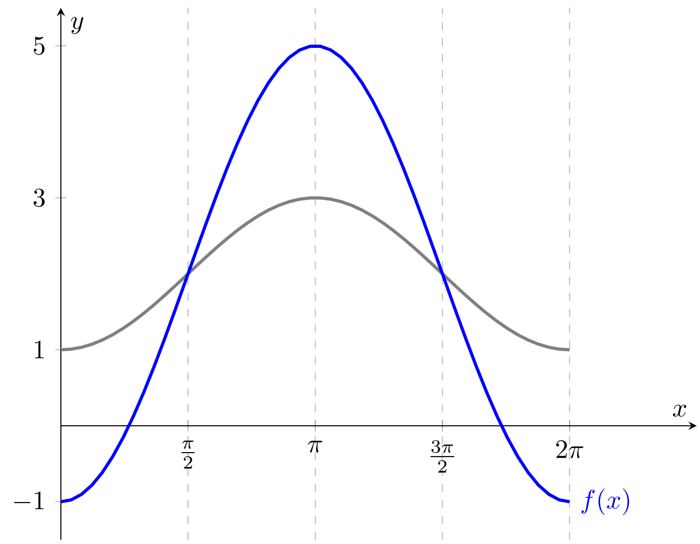
$y = -\cos x$

Then there is a translation by 2 units in the y direction,



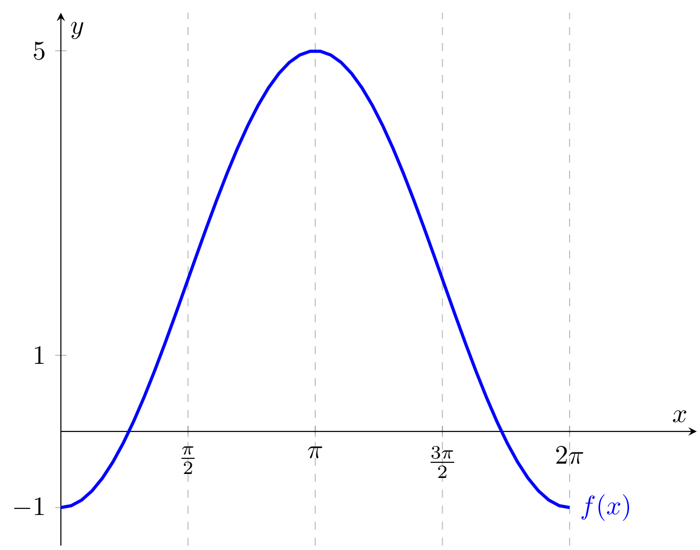
$y = -\cos x + 2$

Finally there is a stretch in the y direction by a stretch factor of 3,



Stretch in the y direction by a stretch factor of 3

Therefore, the graph of $y = f(x)$ is,



$y = f(x)$

Note: Alternatively, you can sketch the graph by first creating a table of values. It is advisable to get comfortable with transformations, and with time you should be able to sketch a combined transformation in one step.

2. The function f is defined by $f(x) = \frac{3}{2}\cos 2x + \frac{1}{2}$ for $0 \leq x \leq \pi$. (9709/11/M/J/20 number 4)

   (a) State the range of f.

$$f(x) = \frac{3}{2}\cos 2x + \frac{1}{2}$$

To get the range of f, we have to find both the maximum and minimum point of f. To find the maximum point, replace $\cos 2x$ with 1,

$$f(x) = \frac{3}{2}(1) + \frac{1}{2}$$
$$f(x) = 2$$

Therefore, the maximum point of f is,

$$2$$

To find the minimum point, replace $\cos 2x$ with -1,

$$f(x) = \frac{3}{2}(-1) + \frac{1}{2}$$
$$f(x) = -1$$

Therefore, the minimum point of f is,

$$-1$$

Therefore, the range of f is,

$$-1 \leq f(x) \leq 2$$

A function g is such that $g(x) = f(x) + k$, where k is a positive constant. The x-axis is a tangent to the curve $y = g(x)$.

   (b) State the value of k and hence describe fully the transformation that maps the curve $y = f(x)$ on to $y = g(x)$.

For the x-axis to be a tangent to the curve the minimum point has to be 0. To make the minimum point 0, we have to add 1 to our current minimum point -1. Therefore,

$$k = 1$$

The transformation is,

Translation in the y-direction by 1 unit

Therefore, the final answer is,

$$k = 1$$

Translation in the y-direction by 1 unit

(c) State the equation of the curve which is the reflection of $y = f(x)$ in the x-axis. Give your answer in the form $y = a\cos 2x + b$ where a and b are constants.

$$f(x) = \frac{3}{2}\cos 2x + \frac{1}{2}$$

If we reflect the function in the $x-axis$ the function becomes negative i.e f(x) becomes -f(x), therefore, we multiply the function by -1,

$$f(x) = -\left(\frac{3}{2}\cos 2x + \frac{1}{2}\right)$$

$$f(x) = -\frac{3}{2}\cos 2x - \frac{1}{2}$$

Therefore, the final answer is,

$$f(x) = -\frac{3}{2}\cos 2x - \frac{1}{2}$$

3. A curve has equation $y = 3\cos 2x + 2$ for $0 \leq x \leq \pi$. (9709/12/O/N/20 number 11)

(a) State the greatest and least values of y.

$$y = 3\cos 2x + 2$$

To find the greatest value of y, replace $\cos 2x$ with 1,

$$y = 3(1) + 2$$

$$y = 5$$

To find the least value of y, replace $\cos 2x$ with -1,

$$y = 3(-1) + 2$$

$$y = -1$$

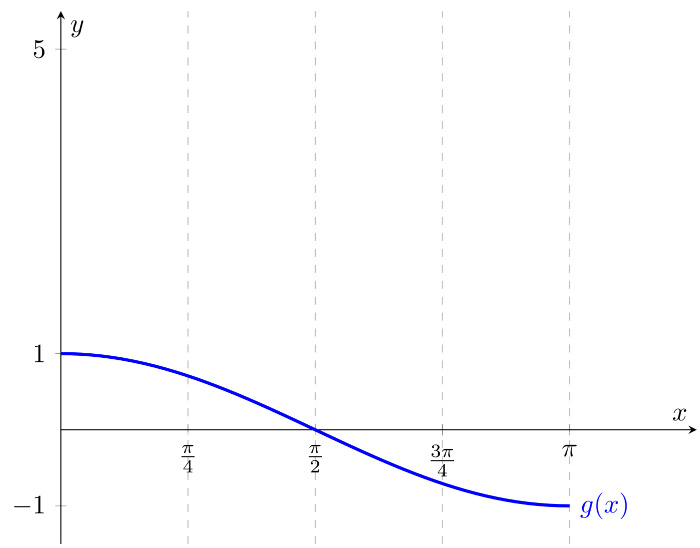
Therefore,

The greatest and least values of y are 5 and -1, respectively.

(b) Sketch the graph of $y = 3\cos 2x + 2$ for $0 \leq x \leq \pi$.

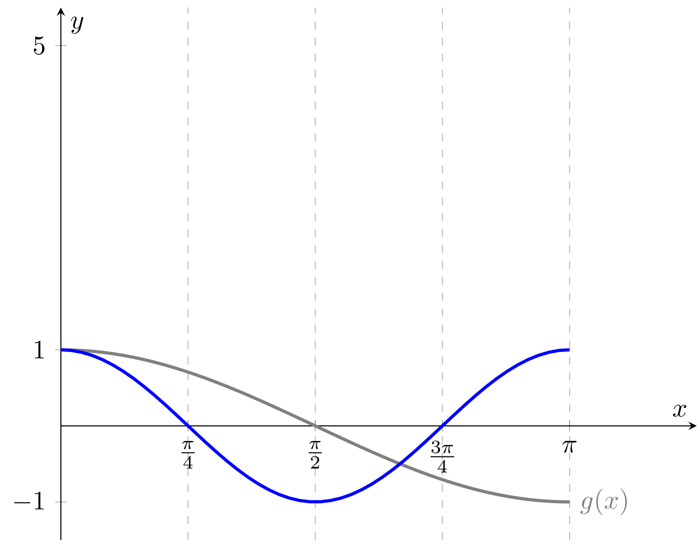
$$y = 3\cos 2x + 2$$

Start by sketching the graph of $y = \cos x$ **for** $0 \leq x \leq \pi$**. We will call this function** $g(x)$**,**



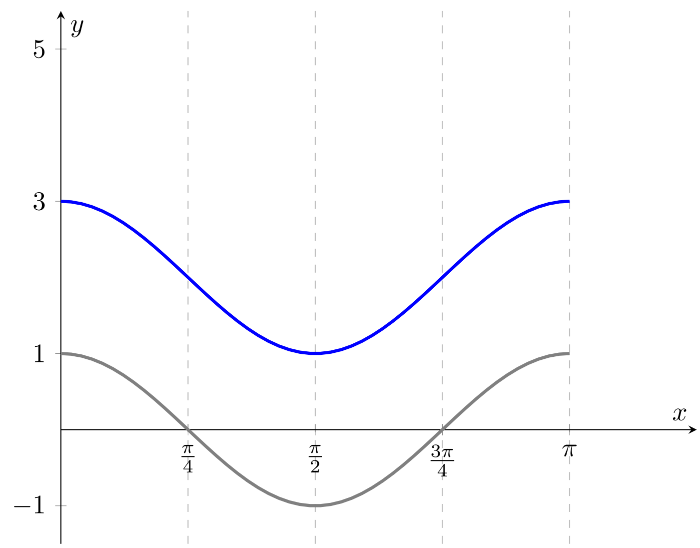
$$g(x) = \cos x$$

From $g(x)$, there is a stretch in the x direction by a stretch factor $\frac{1}{2}$,



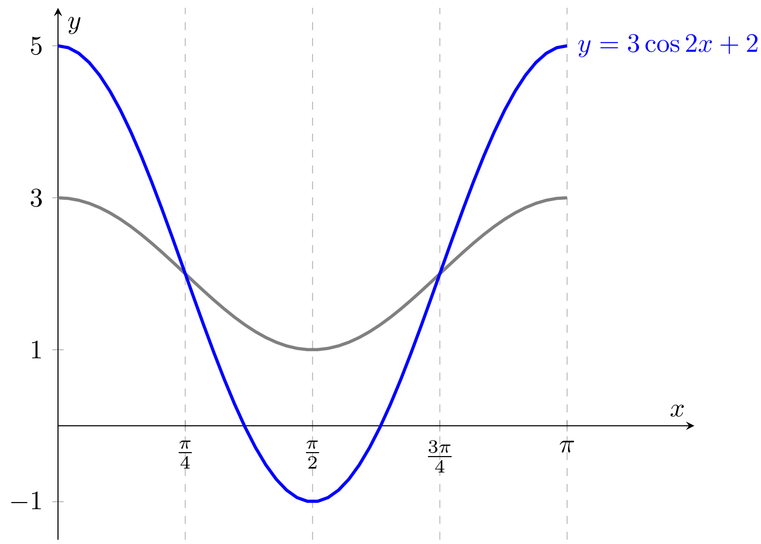
$y = \cos 2x$

Then there is a translation in the y direction by 2 units,



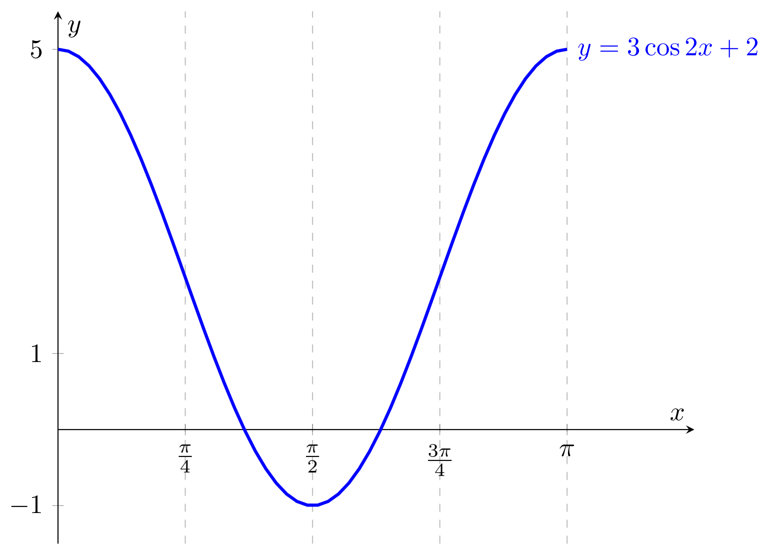
$y = \cos 2x + 2$

Finally, there is a stretch in the y direction by a stretch factor of 3,



$y = 3\cos 2x + 2$

Therefore, the graph of $y = 3\cos 2x + 2$ is,



$y = 3\cos 2x + 2$

(c) By considering the straight line $y = kx$, where k is a constant, state the number of solutions of the equation $3\cos 2x + 2 = kx$ for $0 \leq x \leq \pi$ in each of the following cases.

i. $k = -3$

The line $y = -3x$ does not intersect the graph of $y = 3\cos 2x + 2$, therefore,

$$\text{There are no solutions.}$$

ii. $k = 1$

The line $y = -3x$ intersects the graph of $y = 3\cos 2x + 2$ twice, therefore,

$$\text{There are two solutions.}$$

iii. $k = 3$

The line $y = -3x$ intersects the graph of $y = 3\cos 2x + 2$ once, therefore,

$$\text{There is one solution.}$$

Functions f, g and h are defined for $x \in \mathbb{R}$ by

$$f(x) = 3\cos 2x + 2$$

$$g(x) = f(2x) + 4$$

$$h(x) = 2f\left(x + \frac{1}{2}\pi\right)$$

(d) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ on to $y = g(x)$.

The answer is,

Stretch in the x direction by a stretch factor of $\frac{1}{2}$. Followed by a translation in the y direction by 4 units.

(e) Describe fully a sequence of transformations that maps the graph of $y = f(x)$ on to $y = h(x)$.

The answer is,

Translation in the y direction by $-\frac{1}{2}\pi$ units. Followed by a stretch in the y direction by stretch factor 2.