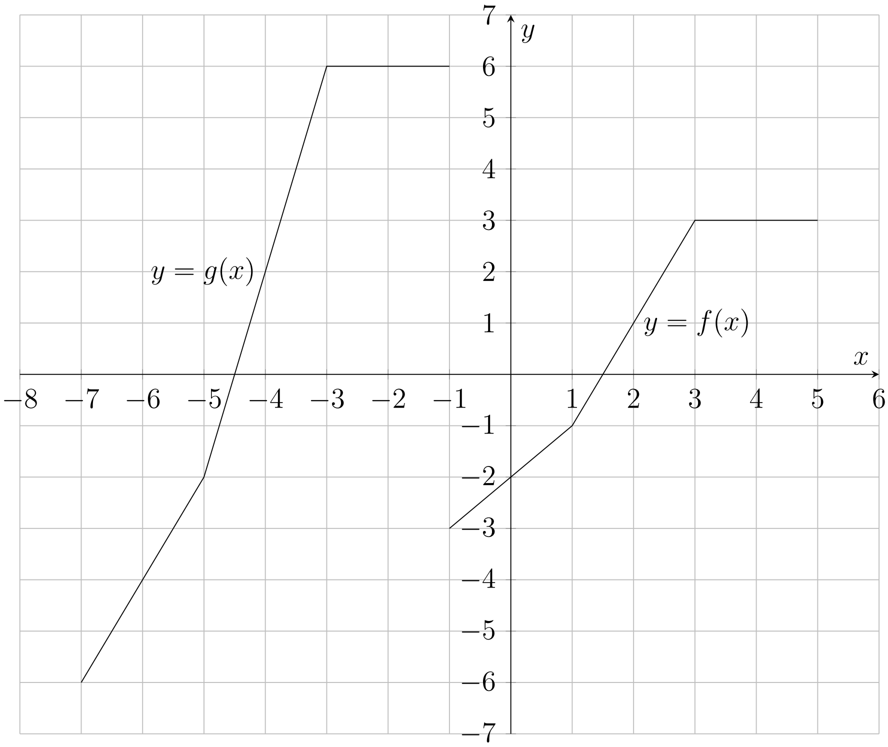
Pure Maths 1

1.2 Functions - Easy

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**AS Level**
Component:	**Pure Mathematics 1**
Topic:	**1.2 Functions**
Difficulty:	**Easy**

Questions

1.



The diagram shows graphs with equations $y = f(x)$ and $y = g(x)$. Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to $y = g(x)$. (9709/11/M/J/23 number 3)

2. The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x - a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

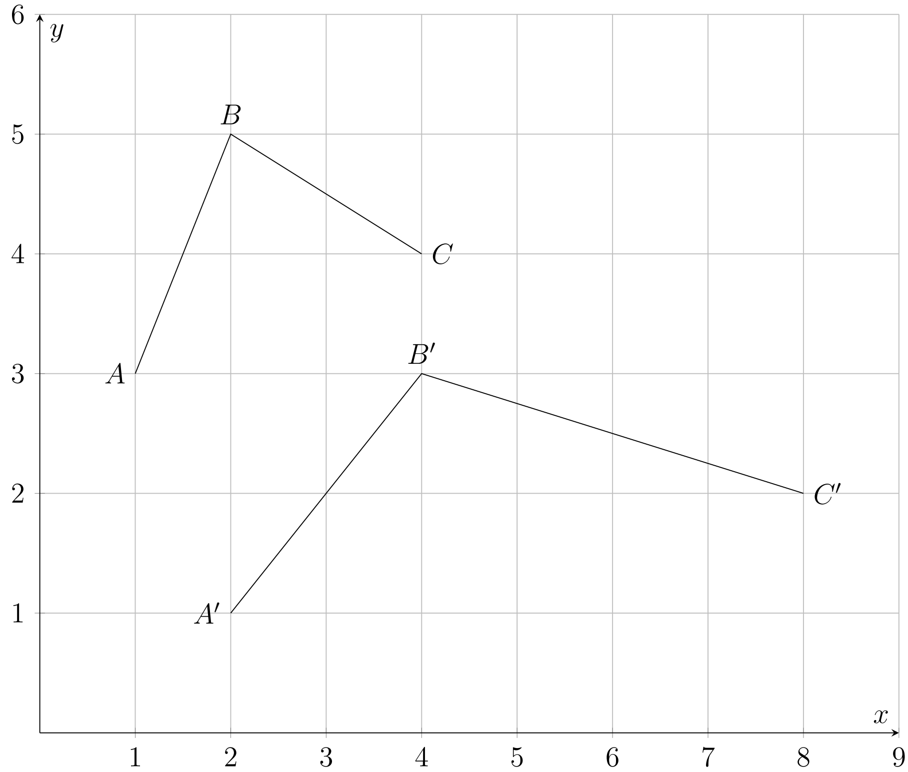
(9709/11/M/J/23 number 8)

(a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b.
For the rest of this question, you should use the value of a which you found in (a).

(b) Find the domain of f^{-1}.

(c) Find an expression of $f^{-}(x)$.

3.



The diagram shows the graph of $y = f(x)$, which consists of the two straight lines AB and BC. The lines $A'B'$ and $B'C'$ form the graph of $y = g(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = f(x)$. State fully the two transformations. (9709/13/M/J/23 number 1)

4. The functions f and g are defined by

$$f(x) = x^2 \text{ for } x \in \mathbb{R}$$

$$g(x) = 2[(x-2)^2 + 3] \text{ for } x \in \mathbb{R}$$

Describe fully a sequence of transformations that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, making clear the order in which the transformations are applied. (9709/12/F/M/22 number 5b)

5. The function f is defined as follows

$$f(x) = \frac{x^2 - 4}{x^2 + 4} \text{ for } x > 2$$

(9709/11/M/J/22 number 6)

(a) Find an expression of $f^{-1}(x)$.

(b) Show that $1 - \frac{8}{x^2+4}$ can be expressed as $\frac{x^2-4}{x^2+4}$ and hence state the range of f.

(c) Explain why the composite function ff cannot be formed.

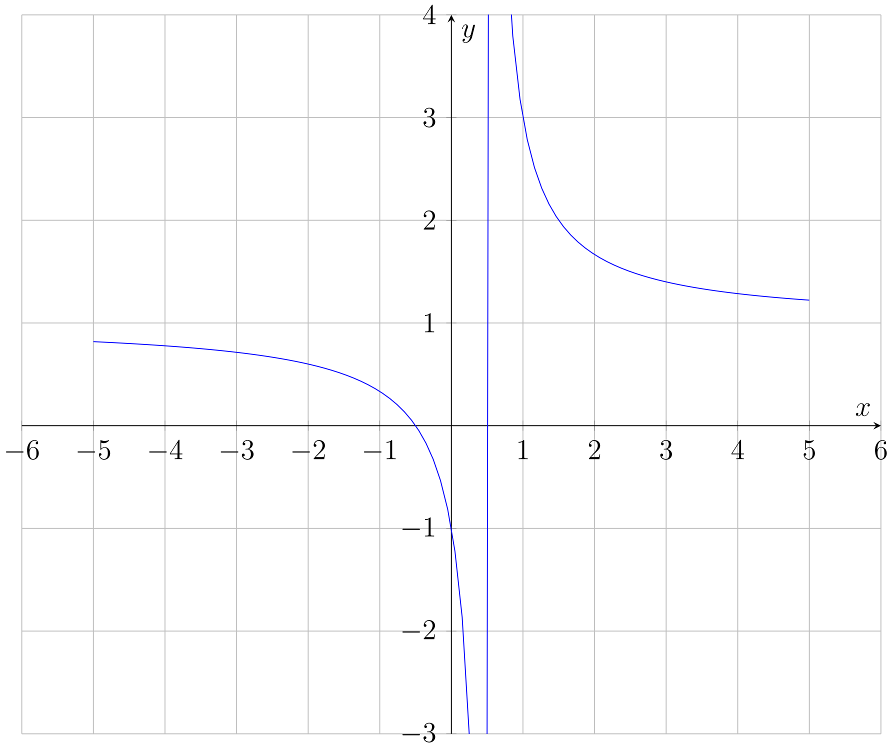
6. Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \text{ for } x \neq \frac{1}{2},$$

$$g(x) = x^2 + 4 \text{ for } x \in \mathbb{R}$$

(9709/12/M/J/22 number 10a-d)

(a)



The diagram shows part of the graph of $y = f(x)$. State the domain of $f^{-1}(x)$.

(b) Find an expression for $f^{-1}(x)$.

(c) Find $gf^{-1}(3)$.

(d) Explain why g^{-1} cannot be found.

7. The function f is defined by $f(x) = 2(x-4)^2 - 9$ for $x < 3$. (9709/13/M/J/22 number 6b-d)

(a) Find the range of f.

(b) Find an expression of $f^{-1}(x)$.
The function g is defined by $g(x) = 2x + 4$ for $x < -1$.

(c) Find and simplify an expression for $fg(x)$.

8. The function f is defined by $f(x) = 2 - \frac{3}{4x-p}$ for $x > \frac{p}{4}$, where p is a constant. (9709/11/O/N/22 number 8bc)

(a) Express $f^{-1}(x)$ in the form $\frac{p}{a} - \frac{b}{cx-d}$, where a, b, c and d.

(b) Hence state the value of p for which $f^{-1}(x) \equiv f(x)$.

9. Functions f and g are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = (x-2)^2 + 5$$

$$g(x) = 2[(x+1)^2 + 5]$$

(9709/11/O/N/22 number 9cd)

(a) Express $g(x)$ in the form $kf(x+h)$, where k and h are integers.

(b) Describe fully the two transformations that have been combined to transform the graph of $y = f(x)$ to the graph of $y = g(x)$.

10. The curve with equation $y = x^2$ is transformed to the curve with equation $y = (x+3)^2 - 4$. Describe fully the transformation(s) involved. (9709/13/O/N/20 number 1b)

11. Functions f and g are defined by

$$f(x) = x + \frac{1}{x} \text{ for } x > 0,$$

$$g(x) = ax + 1 \text{ for } x \in \mathbb{R}$$

where a is a constant. (9709/12/O/N/22 number 9)

(a) Find an expression of $gf(x)$.

(b) Given that $gf(2) = 11$, find the value of a.

(c) Given that the graph of $y = f(x)$ has a minimum point when $x = 1$, explain whether or not f has an inverse.

It is given instead that $a = 5$.

(d) Find and simplify an expression of $g^{-1}f(x)$.

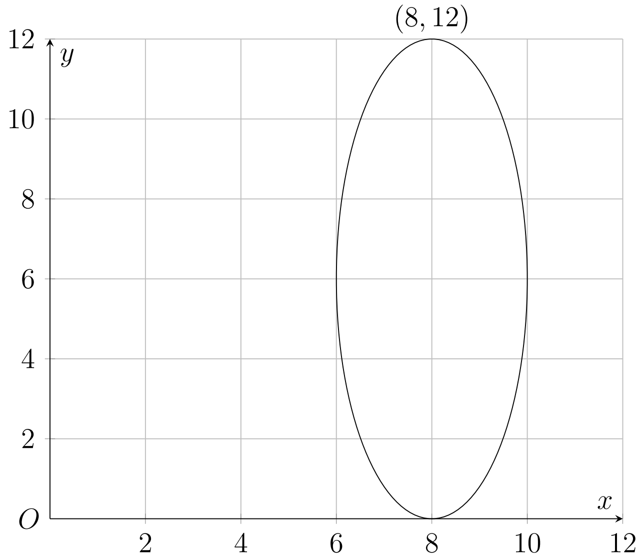
(e) Explain why the composite function fg cannot be formed.

12. The function f is defined by $f(x) = -2(x+2)^2 - 5$ for $x < -3$. (9709/13/O/N/22 number 2bc)

(a) Find the range of f.

(b) Find an expression of $f^{-1}(x)$.

13.



The diagram show a curve which has a maximum point at $(8, 12)$ and a minimum point at $(8, 0)$. The curve is the result of applying a combination of two transformations to a circle. The first transformation applied is a translation of $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$. The second transformation applied is a stretch in the y-direction. (9709/13/O/N/22 number 5)

(a) State the scale factor of the stretch.

(b) State the radius of the original circle.

(c) State the coordinates of the centre of the circle after the translation has been completed but before the stretch is applied.

(d) State the coordinates of the centre of the original circle.

14. Functions f and g are defined as follows:

$$f(x) = x^2 + 2x + 3 \text{ for } x \leq -1,$$

$$g(x) = 2x + 1 \text{ for } x \geq -1$$

(9709/12/F/M/21 number 7)

(a) Express $f(x)$ in the form $(x + a)^2 + b$ and state the range of f.

(b) Find an expression of $f^{-1}(x)$.

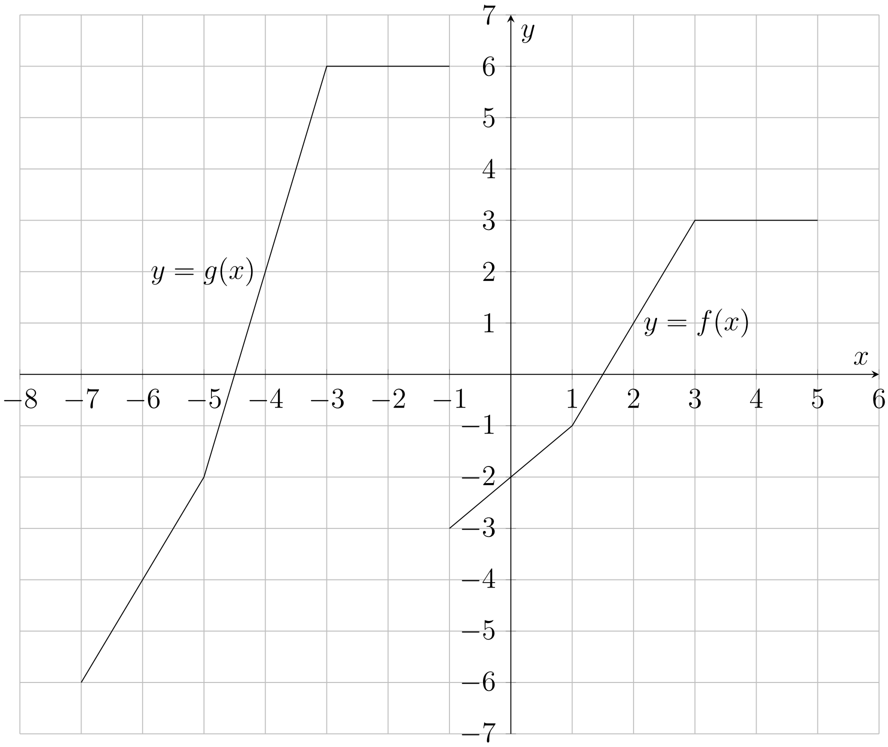
(c) Solve the equation $gf(x) = 13$.

15. (a) The curve with equation $y = x^2 + 2x - 5$ is translated by $\binom{-1}{3}$. (9709/13/M/J/22 number 4) Find the equation of the translated curve, giving your answer in the form $y = ax^2 + bx + c$.

(b) The curve with equation $y = x^2 + 2x - 5$ is transformed to a curve with equation $y = 4x^2 + 4x - 5$. Describe fully the single transformation that has been applied.

Answers

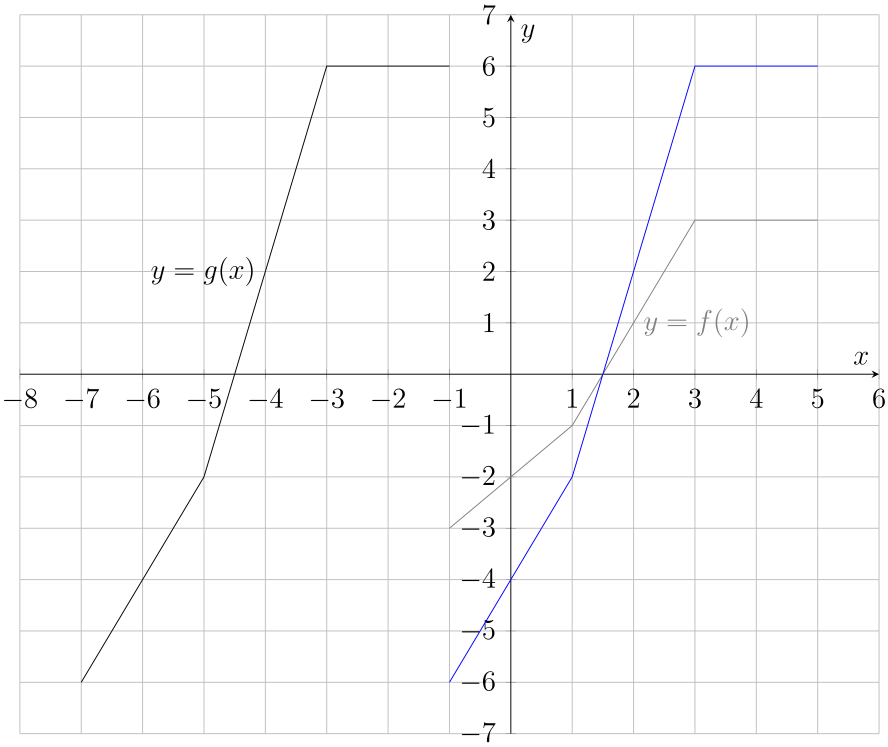
1.



The diagram shows graphs with equations $y = f(x)$ and $y = g(x)$. Describe fully a sequence of two transformations which transforms the graph of $y = f(x)$ to $y = g(x)$. (9709/11/M/J/23 number 3)

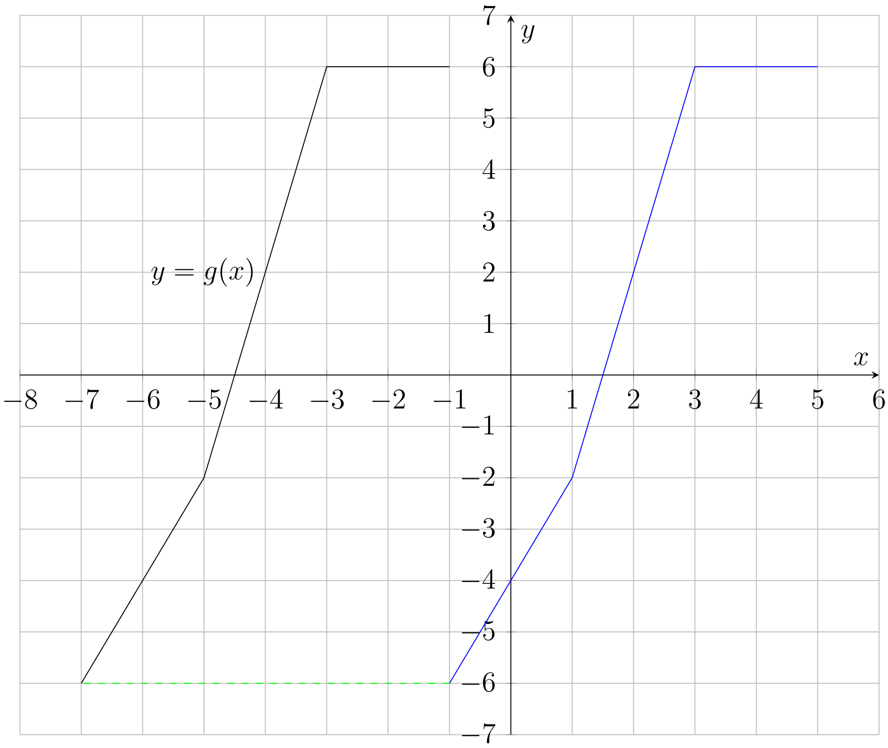
Notice how $y = g(x)$ is vertically longer than $y = f(x)$, this means that we have stretch in the y-direction. To find the stretch factor, count the number of boxes that the two graphs take up in the y-direction. $y = f(x)$ takes up 6 boxes. $y = g(x)$ takes up 12 boxes.

Since the total number of boxes doubles from $f(x)$ to $g(x)$ it means our stretch factor in the y-direction is 2. Multiply the y-values of $f(x)$ by 2 to get the stretched graph,



After the stretch, you can see that the graph of $g(x)$ is to the left of our stretched $y = f(x)$ graph. This is a translation in the x-direction. Count the number of boxes between the two graphs in the x-direction. There are 6 boxes between $f(x)$ and $g(x)$. In this case, one box represents one unit.

Therefore, this is a translation by -6 units in the x-direction,



Therefore, the final answer is,

Stretch in the y-direction, with a stretch factor of 2. Followed by a translation in the x-direction by -6 units.

2. The functions f and g are defined as follows, where a and b are constants.

$$f(x) = 1 + \frac{2a}{x-a} \text{ for } x > a$$

$$g(x) = bx - 2 \text{ for } x \in \mathbb{R}$$

(9709/11/M/J/23 number 8)

(a) Given that $f(7) = \frac{5}{2}$ and $gf(5) = 4$, find the values of a and b.

$$f(7) = \frac{5}{2} \quad gf(5) = 4$$

Let's start by finding $f(7)$. Substitute x with 7 in $f(x)$,

$$f(x) = 1 + \frac{2a}{x-a}$$

$$f(7) = 1 + \frac{2a}{7-a}$$

Equate $f(7)$ to $\frac{5}{2}$,

$$1+\frac{2a}{7-a}=\frac{5}{2}$$

Now we will solve for a. Multiply through by $2(7-a)$ to get rid of the denominators,

$$2(7-a)\times 1+\frac{2a}{7-a}\times 2(7-a)=\frac{5}{2}\times 2(7-a)$$

Expand the brackets and simplify,

$$14-2a+4a=35-5a$$

Group like terms,

$$-2a+4a+5a=35-14$$

Simplify,

$$7a=21$$

Divide both sides by 7,

$$a=3$$

Now let's evaluate $gf(5)=4$. Let's start by evaluating $f(5)$. Substitute 5 into $f(x)$,

$$f(x)=1+\frac{2a}{x-a}$$

$$f(5)=1+\frac{2a}{5-a}$$

Substitute $f(5)$ into $g(x)$,

$$g(x)=bx-2$$

$$g(f(5))=b\left(1+\frac{2a}{5-a}\right)-2$$

Expand the bracket,

$$gf(5)=b+\frac{2ab}{5-a}-2$$

Equate $gf(5)$ to 4,

$$b+\frac{2ab}{5-a}-2=4$$

Earlier we deduced that $a=3$. Substitute a with 3,

$$b+\frac{2(3)b}{5-3}-2=4$$

Simplify,

$$b + \frac{6b}{2} - 2 = 4$$
$$b + 3b - 2 = 4$$
$$4b - 2 = 4$$

Make b the subject of the formula,

$$4b = 6$$
$$b = \frac{3}{2}$$

Therefore, the final answer is,

$$a = 3, \quad b = \frac{3}{2}$$

For the rest of this question, you should use the value of a which you found in (a).

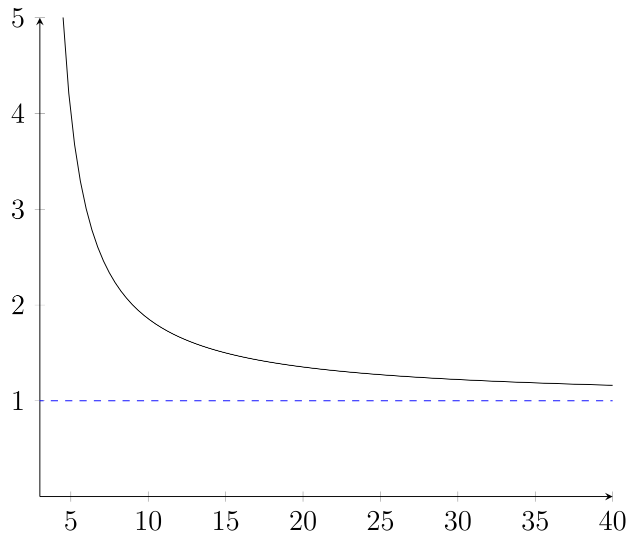
(b) Find the domain of f^{-1}.

$$f(x) = 1 + \frac{2a}{x - a} \text{ for } x > a$$

Let's substitute a with 3,

$$f(x) = 1 + \frac{2(3)}{x - 3} \text{ for } x > 3$$
$$f(x) = 1 + \frac{6}{x - 3} \text{ for } x > 3$$

The domain of the inverse is the range of the original function. So let's find the range of the original function. Let's sketch the graph of $y = f(x)$,



From the sketch we can tell that the range of the function is,

$$y > 1$$

This means that the domain of the inverse is,

$$x > 1$$

Therefore, the final answer is,

$$x > 1$$

(c) Find an expression of $f^{-}(x)$.

$$f(x) = 1 + \frac{2a}{x-a}$$

Let's substitute a with 3,

$$f(x) = 1\frac{2(3)}{x-3}$$

$$f(x) = 1 + \frac{6}{x-3}$$

Now substitute $f(x)$ with y,

$$y = 1 + \frac{6}{x-3}$$

To find $f^{-1}(x)$ we have to make x the subject of the formula. Start by multiplying through by $x-3$ to get rid of the denominator,

$$y(x-3) = 1(x-3) + 6$$

Expand the brackets,

$$xy - 3 = x - 3 + 6$$

$$xy - 3 = x + 3$$

Group like terms,

$$xy - x = 3 + 6$$

$$xy - x = 9$$

Factor out x on the left hand side,

$$x(y-1) = 9$$

Divide both sides by $y-1$ to make x the subject of the formula,

$$x = \frac{9}{y-1}$$

This means that $f^{-1}(x)$ is,

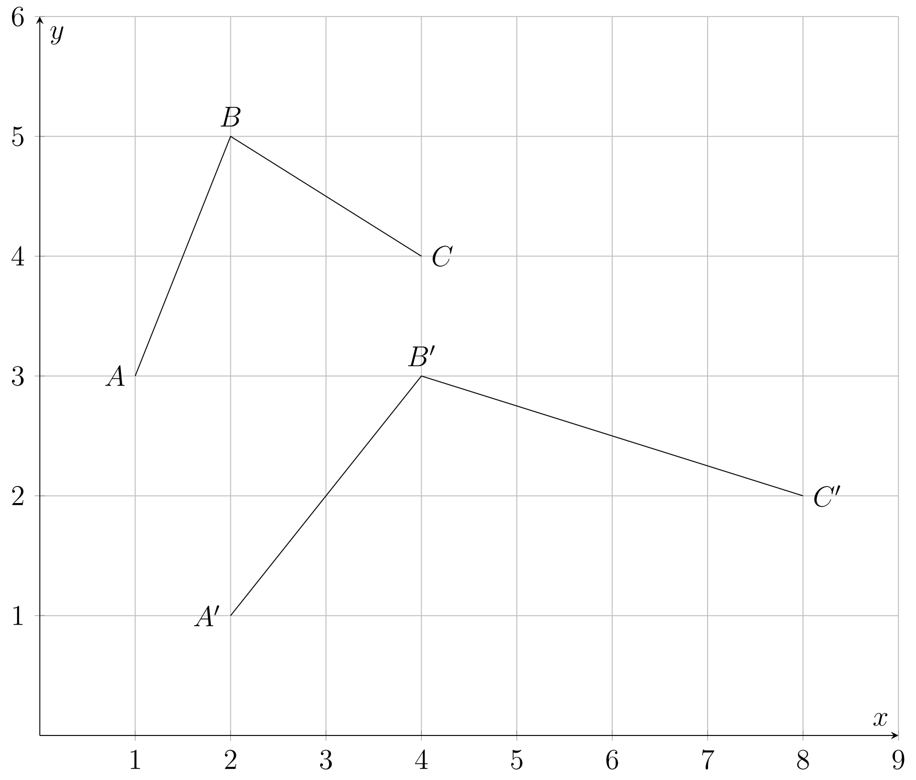
$$f^{-1}(x) = \frac{9}{x-1}$$

Note: Make sure to change y back to x in the inverse function.

Therefore, the final answer is,

$$f^{-1}(x) = \frac{9}{x-1}$$

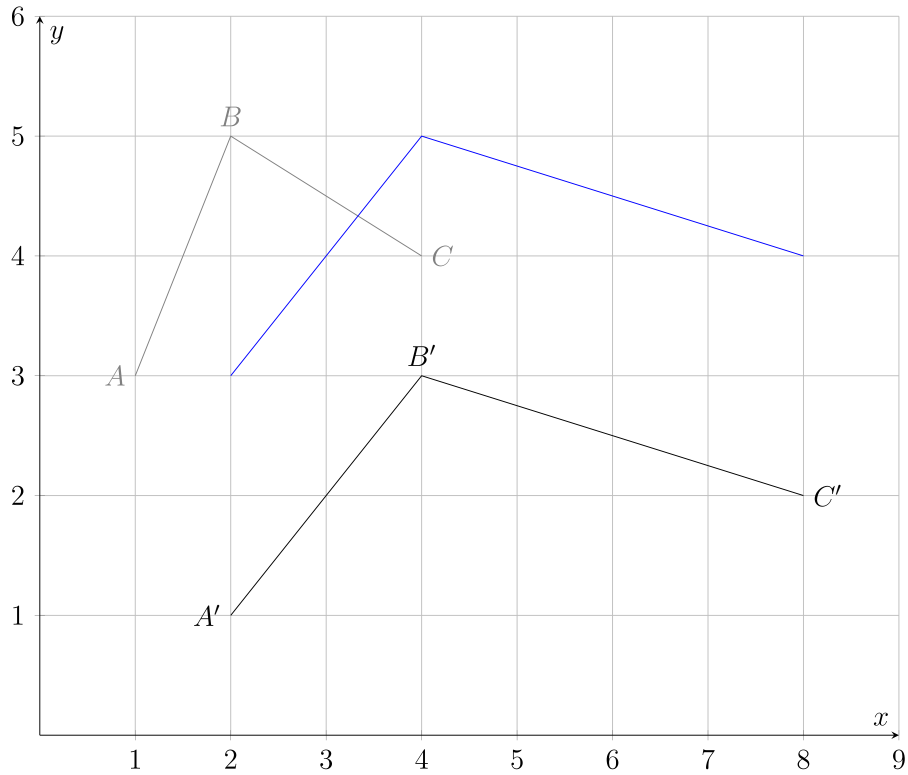
3.



The diagram shows the graph of $y = f(x)$, which consists of the two straight lines AB and BC. The lines $A'B'$ and $B'C'$ form the graph of $y = g(x)$, which is the result of applying a sequence of two transformations, in either order, to $y = f(x)$. State fully the two transformations. (9709/13/M/J/23 number 1)

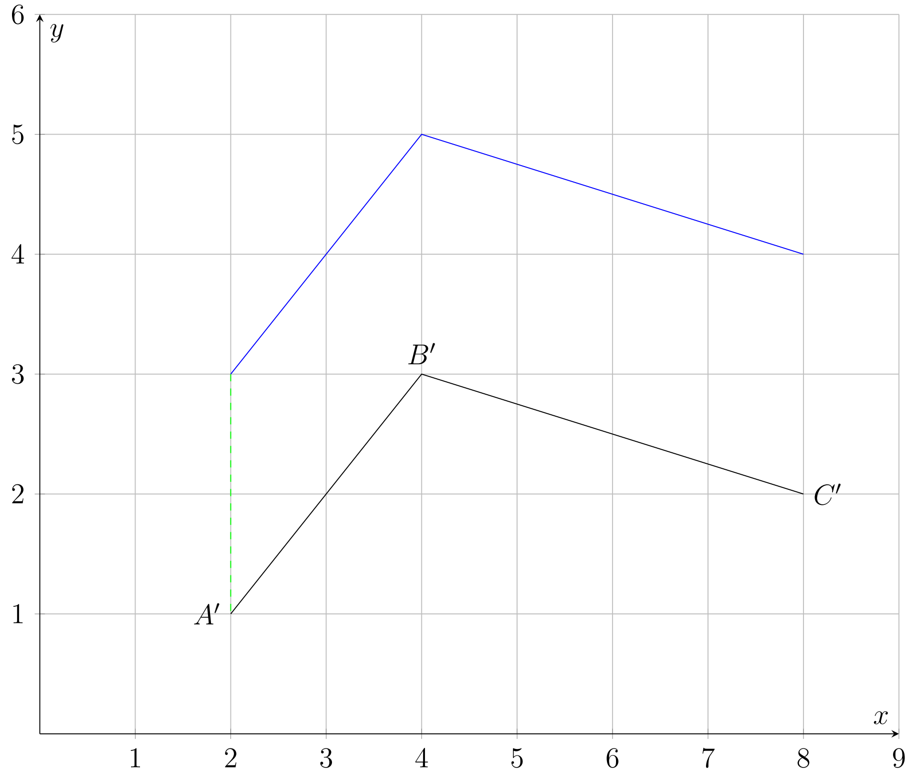
Notice how $y = g(x)$ is horizontally longer than $y = f(x)$, this means that we have stretch in the x-direction. To find the stretch factor, count the number of boxes that the two graphs take up in the x-direction. $y = f(x)$ takes up 3 boxes. $y = g(x)$ takes up 6 boxes.

Since the total number of boxes doubles from $f(x)$ to $g(x)$ it means our stretch factor is 2. Multiply the x-coordinates by 2 to stretch the graph of $f(x)$ in the x-axis by stretch factor 2,



After the stretch, you can see that $g(x)$ lies directly below $f(x)$. This is a translation in the y-direction. Count the number of boxes between the two graphs in the y-direction. There are 2 boxes between $f(x)$ and $g(x)$. In this case, one box represents one unit.

Therefore, this is a translation by -2 units in the y-direction,



Therefore, the final answer is,

Stretch in the x-direction, with a stretch factor of 2. Followed by a translation in the y-direction by -2 units.

4. The functions f and g are defined by

$$f(x) = x^2 \text{ for } x \in \mathbb{R}$$

$$g(x) = 2[(x-2)^2 + 3] \text{ for } x \in \mathbb{R}$$

Describe fully a sequence of transformations that maps the graph of $y = f(x)$ onto the graph of $y = g(x)$, making clear the order in which the transformations are applied. (9709/12/F/M/22 number 5b)

$$g(x) = 2[(x - 2)^2 + 3]$$

The -2 represents a translation in the x-direction by 2 units.

Note: For transformations in the x-direction we invert the sign. In the case of a translation, a negative sign becomes a positive sign and vice versa. In the case of a stretch, we take the reciprocal i.e $f\left(\frac{1}{a}x\right)$ has a stretch factor of a.

$$g(x) = 2[(x-2)^2 + 3]$$

The 3 represents a translation in the y-direction by 3 units.

Note: For transformations in the y-direction we DO NOT invert the sign.

$$g(x) = 2[(x-2)^2 + 3]$$

The 2 represents a stretch in the y-direction by a stretch factor of 2.

Therefore, the final answer would be,

Translation with translation vector, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Followed by a stretch in the y-direction by a stretch factor of 2.

5. The function f is defined as follows

$$f(x) = \frac{x^2 - 4}{x^2 + 4} \text{ for } x > 2$$

(9709/11/M/J/22 number 6)

(a) Find an expression of $f^{-1}(x)$.

$$f(x) = \frac{x^2 - 4}{x^2 + 4}$$

Substitute $f(x)$ with y,

$$y = \frac{x^2 - 4}{x^2 + 4}$$

To find $f^{-1}(x)$ we have to make x the subject of the formula. Start by multiplying through by $x^2 + 4$ to get rid of the denominator,

$$y\left(x^2 + 4\right) = x^2 - 4$$

Expand the brackets,

$$x^2y + 4y = x^2 - 4$$

Put the terms containing x^2 on one side,

$$x^2y - x^2 = -4y - 4$$

Factor out x^2 on the left hand side,

$$x^2(y-1) = -4y - 4$$

Divide both sides by $y - 1$ to make x^2 the subject of the formula,

$$x^2 = \frac{-4y - 4}{y - 1}$$

Take the square root of both sides,

$$x = \pm\sqrt{\frac{-4y - 4}{y - 1}}$$

This means that $f^{-1}(x)$ is,

$$f^{-1}(x) = \sqrt{\frac{-4x-4}{x-1}}$$

Note: Your final answer should not have $\pm$. The sign is determined by the domain of the original function. If the domain has $>$ sign, then the inverse is positive. If the domain has a $<$ sign, the inverse is negative.

Therefore, the final answer is,

$$f^{-1}(x) = \sqrt{\frac{-4x-4}{x-1}}$$

(b) Show that $1 - \frac{8}{x^2+4}$ can be expressed as $\frac{x^2-4}{x^2+4}$ and hence state the range of f.

$$1 - \frac{8}{x^2+4}$$

Find the common denominator of the two terms,

$$x^2 + 4$$

Now let's combine the two terms into one,

$$\frac{x^2+4-8}{x^2+4}$$

Simplify the numerator,

$$\frac{x^2-4}{x^2+4}$$

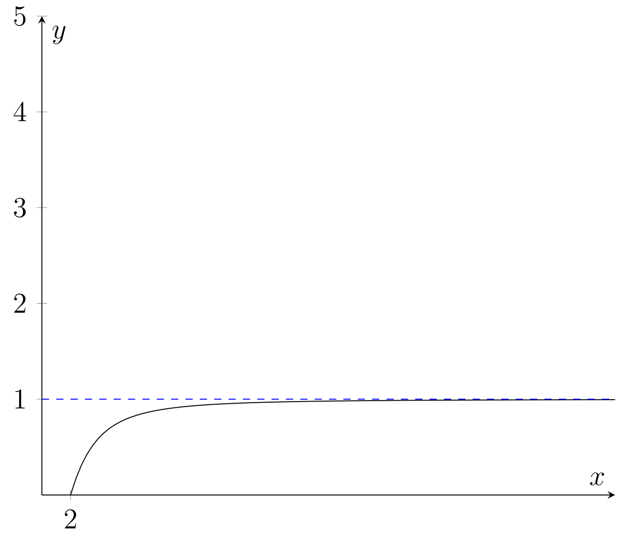
Therefore, we have expressed $1 - \frac{8}{x^2+4}$ as,

$$\frac{x^2-4}{x^2+4}$$

(c) Explain why the composite function ff cannot be formed.

For a composite function to be formed, the range of the inner function must fit into the domain of the outer function i.e range of inner function should be a subset of domain of outer function. If this condition is not satisfied, a composite function cannot be formed. In this case, the inner function is f and the outer function is also f. So we need to check that the range of f fits completely into the domain of f.

Let's sketch the graph of $y = f(x)$,



From the graph we can tell that the range of f is,

$$0 < y < 1$$

The domain of f is given as,

$$x > 2$$

This shows that the range of f lies completely outside of the domain of f.

Therefore, the final answer is,

ff cannot be formed because the range of f does not include the whole of the domain of f

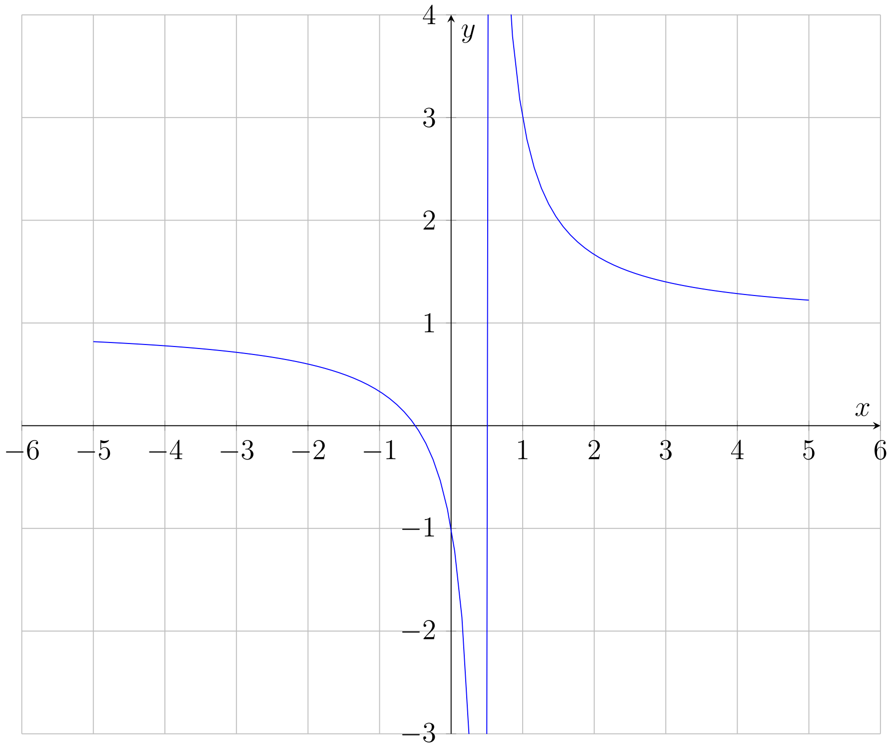
6. Functions f and g are defined as follows:

$$f(x) = \frac{2x+1}{2x-1} \text{ for } x \neq \frac{1}{2},$$

$$g(x) = x^2 + 4 \text{ for } x \in \mathbb{R}$$

(9709/12/M/J/22 number 10a-d)

(a)



The diagram shows part of the graph of $y = f(x)$. State the domain of $f^{-1}(x)$.

The domain of the inverse function is the range of the original function. From the diagram, we can tell that the range of f is,

$$y \in \mathbb{R},\ y \neq 1$$

This means that the the domain of f^{-1} is,

$$x \in \mathbb{R},\ x \neq 1$$

Note: The domain is always written in terms of x.

Therefore, the final answer is,

$$x \in \mathbb{R},\ x \neq 1$$

(b) Find an expression for $f^{-1}(x)$.

$$f(x) = \frac{2x+1}{2x-1}$$

Substitute $f(x)$ with y,

$$y = \frac{2x+1}{2x-1}$$

To find $f^{-1}(x)$ we have to make x the subject of the formula. Start by multiplying through by $2x-1$ to get rid of the denominator,

$$y(2x-1) = 2x+1$$

Expand the brackets,

$$2xy - y = 2x + 1$$

Put the terms containing x on one side,

$$2xy - 2x = y + 1$$

Factor out x on the left hand side,

$$x(2y-2) = y+1$$

Divide both sides by $2y-2$ to make x the subject of the formula,

$$x = \frac{y+1}{2y-2}$$

This means that $f^{-1}(x)$ is,

$$f^{-1}(x) = \frac{x+1}{2x-2}$$

Therefore, the final answer is,

$$f^{-1}(x) = \frac{x+1}{2x-2}$$

(c) Find $gf^{-1}(3)$.

$$g(x) = x^2 + 4 \qquad f^{-1}(x) = \frac{x+1}{2x-2}$$

Let's start by finding $f^{-1}(3)$. Substitute 3 into f^{-1},

$$f^{-1}(3) = \frac{3+1}{2(3)-2}$$

Simplify,

$$f^{-1}(3) = 1$$

Now substitute your value for $f^{-1}(3)$ into into $g(x)$,

$$g(x) = x^2 + 4$$

$$g(f^{-1}(3)) = 1^2 + 4$$

Simplify,

$$gf^{-1}(3) = 5$$

Therefore, the final answer is,

$$gf^{-1}(3) = 5$$

(d) Explain why g^{-1} cannot be found.

For a function to have an inverse, it must be a one to one function. In this case g is not one to one hence g^{-1} cannot be formed.

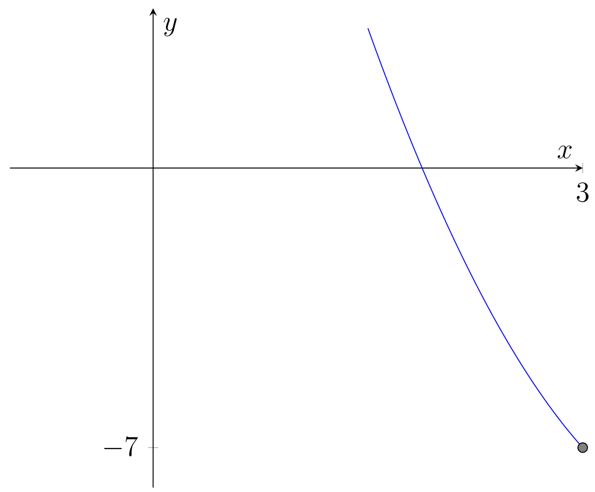
Therefore, the final answer is,

g is not a one to one function, hence g^{-1} cannot be formed.

7. The function f is defined by $f(x) = 2(x-4)^2 - 9$ for $x < 3$. (9709/13/M/J/22 number 6b-d)

(a) Find the range of f.

To find the range of f let's sketch the graph of $y = f(x)$,



From the graph, we can tell that the range of f is,

$$y > -7$$

Therefore, the final answer is,

$$y > -7$$

(b) Find an expression of $f^{-1}(x)$.

$$f(x) = 2(x-4)^2 - 9$$

Start by substituting $f(x)$ with y,

$$y = 2(x-4)^2 - 9$$

Add 9 to both sides,

$$y + 9 = 2(x - 4)^2$$

Divide both sides by 2,

$$\frac{y+9}{2} = (x-4)^2$$

Take the square root of both sides,

$$\pm\sqrt{\frac{y+9}{2}} = x - 4$$

Add 4 to both sides,

$$x = 4 \pm \sqrt{\frac{y+9}{2}}$$

Since the domain of f has a $<$ sign, we will take the negative sign,

$$x = 4 - \sqrt{\frac{y+9}{2}}$$

Therefore, the final answer is,

$$f^{-1}(x) = 4 - \sqrt{\frac{x+9}{2}}$$

The function g is defined by $g(x) = 2x + 4$ for $x < -1$.

(c) Find and simplify an expression for $fg(x)$.

$$f(x) = 2(x-4)^2 - 9 \qquad g(x) = 2x + 4$$

Substitute the x in $f(x)$ with $g(x)$,

$$f(x) = 2(x-4)^2 - 9$$
$$f(g(x)) = 2(2x + 4 - 4)^2 - 9$$

Simplify,

$$fg(x) = 2(2x)^2 - 9$$
$$fg(x) = 2\left(4x^2\right) - 9$$
$$fg(x) = 8x^2 - 9$$

Therefore, the final answer is,

$$fg(x) = 8x^2 - 9$$

8. The function f is defined by $f(x) = 2 - \frac{3}{4x-p}$ for $x > \frac{p}{4}$, where p is a constant. (9709/11/O/N/22 number 8bc)

   (a) Express $f^{-1}(x)$ in the form $\frac{p}{a} - \frac{b}{cx-d}$, where a, b, c and d.

$$f(x) = 2 - \frac{3}{4x-p}$$

Substitute $f(x)$ with y,

$$y = 2 - \frac{3}{4x - p}$$

To find $f^{-1}(x)$ we have to make x the subject of the formula. Start by multiplying through by $4x - p$ to get rid of the denominator,

$$y(4x - p) = 2(4x - p) - 3$$

Expand the brackets,

$$4xy - py = 8x - 2p - 3$$

Put the terms containing x on one side,

$$4xy - 8x = py - 2p - 3$$

Factor out x on the left hand side,

$$x(4y - 8) = py - 2p - 3$$

Divide both sides by $4y - 8$ to make x the subject of the formula,

$$x = \frac{py - 2p - 3}{4y - 8}$$

Now we have to express it in the form $\frac{p}{a} - \frac{b}{cx-d}$. Let's start by factoring out a p in the first 2 terms in the numerator,

$$x = \frac{p(y - 2) - 3}{4y - 8}$$

There are now two terms in the numerator. Distribute the denominator to separate them,

$$x = \frac{p(y - 2)}{4y - 8} - \frac{3}{4y - 8}$$

In the first term, factor out 4 in the denominator,

$$x = \frac{p(y - 2)}{4(y - 2)} - \frac{3}{4y - 8}$$

Cancel out $y - 2$ in the first term,

$$x = \frac{p}{4} - \frac{3}{4y - 8}$$

This means that $f^{-1}(x)$ is,

$$f^{-1}(x) = \frac{p}{4} - \frac{3}{4x - 8}$$

Therefore, the final answer is,

$$f^{-1}(x) = \frac{p}{4} - \frac{3}{4x - 8}$$

(b) Hence state the value of p for which $f^{-1}(x) \equiv f(x)$.

$$f^{-1}(x) = \frac{p}{4} - \frac{3}{4x - 8} \qquad f(x) = 2 - \frac{3}{4x - p}$$

Notice the similarities between f^{-1} and f,

$$f^{-1}(x) = \frac{p}{4} - \frac{3}{4x - 8} \qquad f(x) = 2 - \frac{3}{4x - p}$$

Just by inspecting the two equations, we can tell that for $f^{-1}(x) \equiv f(x)$,

$$p = 8$$

Therefore, the final answer is,

$$p = 8$$

9. Functions f and g are both defined for $x \in \mathbb{R}$ and are given by

$$f(x) = (x - 2)^2 + 5$$

$$g(x) = 2[(x + 1)^2 + 5]$$

(9709/11/O/N/22 number 9cd)

(a) Express $g(x)$ in the form $kf(x + h)$, where k and h are integers.

$$f(x) = (x - 2)^2 + 5 \qquad g(x) = 2[(x + 1)^2 + 5]$$

If we compare the two functions, we will notice a 2 on g. This represents a stretch in the y-direction. We also notice that -2 has changed to 1. This is a translation. To get the number of units we simply substract the two numbers $(1 - (-2))$ to get 3. Now to answer the question,

$$g(x) = 2f(x + 3)$$

Therefore, the final answer is,

$$g(x) = 2f(x + 3)$$

(b) Describe fully the two transformations that have been combined to transform the graph of $y = f(x)$ to the graph of $y = g(x)$.

Translation in the x-direction by -3 units. Followed by a stretch in the y-direction by a stretch factor of 2

10. The curve with equation $y = x^2$ is transformed to the curve with equation $y = (x + 3)^2 - 4$. Describe fully the transformation(s) involved. (9709/13/O/N/20 number 1b)

$$y = (x + 3)^2 - 4$$

The 3 represents a translation in the x-direction by -3 units. The -4 represents a translation in the y-direction by -4 units.

Therefore, the final answer is,

$$\text{Translation by a translation vector of } \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

11. Functions f and g are defined by

$$f(x) = x + \frac{1}{x} \text{ for } x > 0,$$

$$g(x) = ax + 1 \text{ for } x \in \mathbb{R}$$

where a is a constant. (9709/12/O/N/22 number 9)

(a) Find an expression of $gf(x)$.

$$f(x) = x + \frac{1}{x} \qquad g(x) = ax + 1$$

Substitute x in $g(x)$ with $f(x)$,

$$g(x) = ax + 1$$

$$g(f(x)) = a\left(x + \frac{1}{x}\right) + 1$$

Expand the bracket,

$$gf(x) = ax + \frac{a}{x} + 1$$

Therefore, the final answer is,

$$gf(x) = ax + \frac{a}{x} + 1$$

(b) Given that $gf(2) = 11$, find the value of a.

$$gf(x) = ax + \frac{a}{x} + 1$$

Substitute x in $gf(x)$ with 2,

$$gf(x) = ax + \frac{a}{x} + 1$$

$$gf(2) = a2 + \frac{a}{2} + 1$$

Simplify,

$$gf(2) = 2a + \frac{a}{2} + 1$$

$$gf(2) = \frac{5}{2}a + 1$$

Equate $gf(2)$ to 11,

$$\frac{5}{2}a + 1 = 11$$

Make a the subject of the formula,

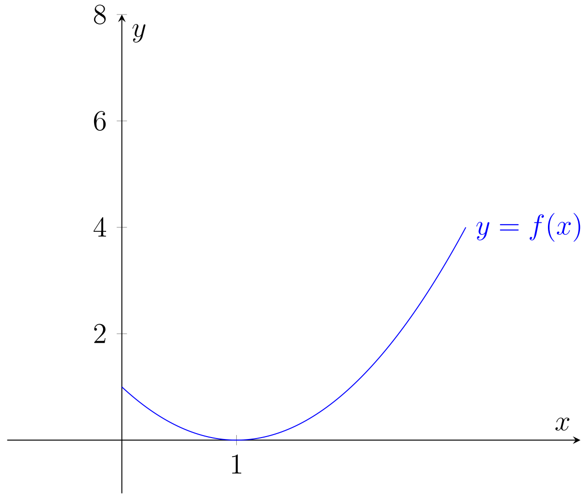
$$\frac{5}{2}a = 10$$

$$\frac{5}{2}a \times \frac{2}{5} = 10 \times \frac{2}{5}$$
$$a = 4$$

Therefore, the final answer is,

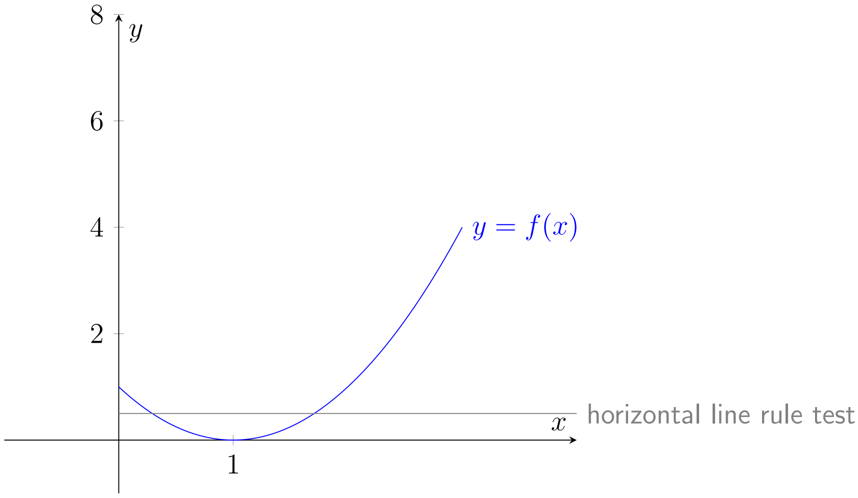
$$a = 4$$

(c) Given that the graph of $y = f(x)$ has a minimum point when $x = 1$, explain whether or not f has an inverse.



Note: The above diagram is not an exact sketch of the function f. Since we are told that it has a minimum point at $x = 1$, so the lowest point on the graph is at $x = 1$. So either side of $x = 1$ the graph will be above the minimum point.

Let's use the horizontal rule test,



The horizontal line crosses the function twice. So f is not a one to one function. We can conclude that f has no inverse.

Therefore, the final answer is,

$$f \text{ has no inverse because it is not a one to one function}$$

It is given instead that $a = 5$.

(d) Find and simplify an expression of $g^{-1}f(x)$.

$$f(x) = x + \frac{1}{x} \qquad g(x) = ax + 1$$

Since a is given as 5, we can write $g(x)$ as,

$$g(x) = 5x + 1$$

Now let's find g^{-1}. Start by substituting $g(x)$ with y,

$$y = 5x + 1$$

Subtract 1 from both sides,

$$5x = y - 1$$

Divide both sides 5,

$$x = \frac{y-1}{5}$$

Therefore, g^{-1} is,

$$g^{-1}(x) = \frac{x-1}{5}$$

Now let's substitute $f(x)$ into $g^{-1}f(x)$,

$$g^{-1}(x) = \frac{x-1}{5}$$

$$g^{-1}(f(x)) = \frac{x+\frac{1}{x}-1}{5}$$

Find the common denominator for the numerator,

$$g^{-1}f(x) = \frac{\frac{x^2+1-x}{x}}{5}$$

Simplify the fraction,

$$g^{-1}f(x) = \frac{x^2+1-x}{5x}$$

Therefore, the final answer is,

$$g^{-1}f(x) = \frac{x^2+1-x}{5x}$$

(e) Explain why the composite function fg cannot be formed.

$$f(x) = x + \frac{1}{x} \text{ for } x > 0 \quad g(x) = 5x + 1 \text{ for } x \in \mathbb{R}$$

For the composite function fg to be possible the range of g should fit inside the domain of f. Since the domain of g is $x \in \mathbb{R}$ the range of g is,

$$y \in \mathbb{R}$$

Now let's compare the range of g to the domain of f,

$$y \in \mathbb{R} \quad x > 0$$

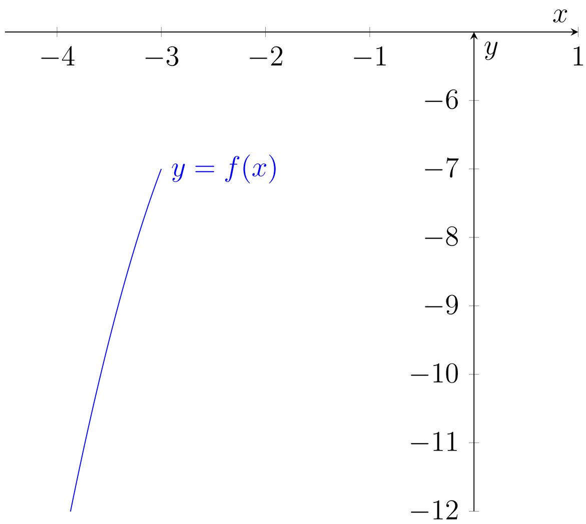
The domain of f does not include the whole range of g. Therefore, the final answer is,

fg cannot be formed because the domain of f does not include the whole range of g

12. The function f is defined by $f(x) = -2(x+2)^2 - 5$ for $x < -3$. (9709/13/O/N/22 number 2bc)

(a) Find the range of f.

Let's sketch the graph of $y = f(x)$,



From the diagram, we can tell that the range of $y = f(x)$ is,

$$y > -7$$

Therefore, the final answer is,

$$y > -7$$

(b) Find an expression of $f^{-1}(x)$.

$$f(x) = -2(x+2)^2 - 5$$

Start by substituting $f(x)$ with y,

$$y = -2(x+2)^2 - 5$$

Add 5 to both sides,

$$y + 5 = -2(x+2)^2$$

Divide both sides by -2,

$$\frac{y+5}{-2} = (x+2)^2$$

Take the square root of both sides,

$$\pm\sqrt{\frac{y+5}{-2}} = x + 2$$

Subtract 2 from both sides,

$$x = -2 \pm \sqrt{\frac{y+5}{-2}}$$

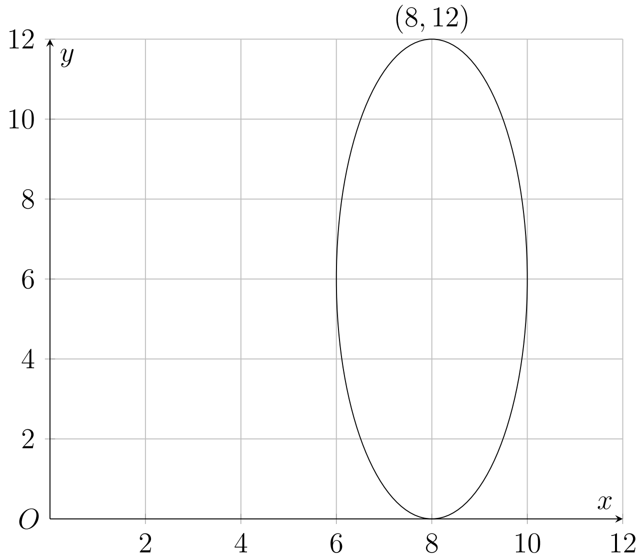
Since the domain of f has a $<$ sign, we will take the negative sign,

$$x = -2 - \sqrt{\frac{y+5}{-2}}$$

Therefore, the final answer is,

$$f^{-1}(x) = -2 - \sqrt{\frac{x+5}{-2}}$$

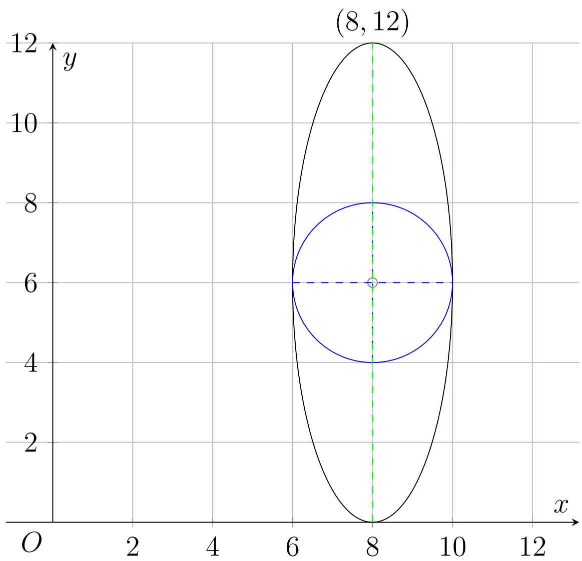
13.



The diagram show a curve which has a maximum point at $(8, 12)$ and a minimum point at $(8, 0)$. The curve is the result of applying a combination of two transformations to a circle. The first transformation applied is a translation of $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$. The second transformation applied is a stretch in the y-direction. (9709/13/O/N/22 number 5)

(a) State the scale factor of the stretch.

The initial function was a circle. Then this circle was translated. After the translation, the circle was stretched in the y-direction. There is no stretch in the x-direction, so the width of the stretched function should give us the radius of the circle. The stretched function takes up a maximum of two boxes in the x-direction. This means the diameter is two boxes, therefore, the radius is one box. If we sketch a circle with a radius of one box, and compare it to the stretched function we notice that, in the y-direction, the circle takes up 2 boxes and the stretched function takes up 6 boxes. The number of boxes tripled. This means the scale factor is 3,



Therefore, the final answer is,

The scale factor is 3

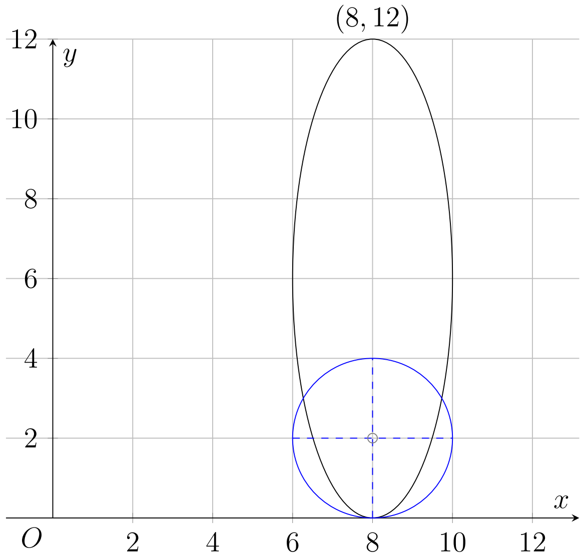
(b) State the radius of the original circle.

We already deduced that the radius of the original circle is 1 box. According to the scale of the diagram, 1 box is equal to 2 units. Therefore, the final answer is,

The radius of the orignal circle is 2

(c) State the coordinates of the centre of the circle after the translation has been completed but before the stretch is applied.

To get the original circle we just undo the stretch in the y-direction. We know that there is a stretch in the y-direction by a stretch factor of 3. To undo this stretch, we apply a stretch in the y-direction with a stretch factor of $\frac{1}{3}$. Multiply the y-coordinates of the stretched function by $\frac{1}{3}$ and plot a knew circle using the resulting coordinates. This will give you the original circle after the translation,



From the diagram above, we can tell that the coordinates of the centre of the circle after the translation before the stretch are,

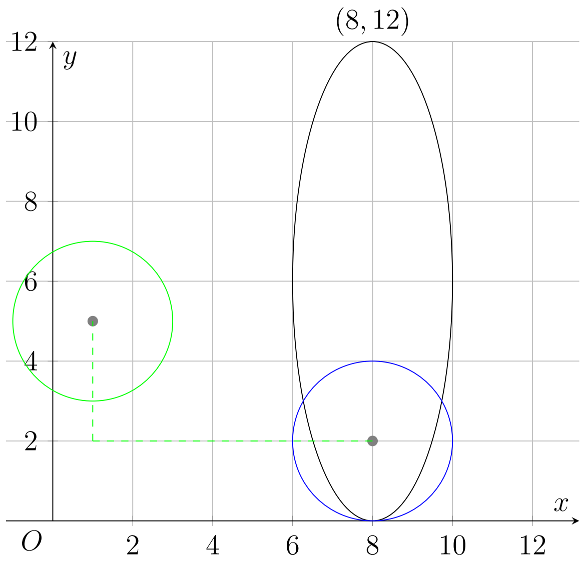
$$(8, 2)$$

Therefore, the final answer is,

$$(8, 2)$$

(d) State the coordinates of the centre of the original circle.

To get the coordinates of the original centre, we have to reverse the translation from the circle before the stretch. The circle was translated using the translation vector $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$**. To reverse this translation we will apply a translation of** $\begin{pmatrix} -7 \\ 3 \end{pmatrix}$**. From there, we can deduce the coordinates of the original centre,**



Therefore, the final answer is,

$$(1, 5)$$

14. Functions f and g are defined as follows:

$$f(x) = x^2 + 2x + 3 \text{ for } x \leq -1,$$

$$g(x) = 2x + 1 \text{ for } x \geq -1$$

(9709/12/F/M/21 number 7)

(a) Express $f(x)$ in the form $(x + a)^2 + b$ and state the range of f.

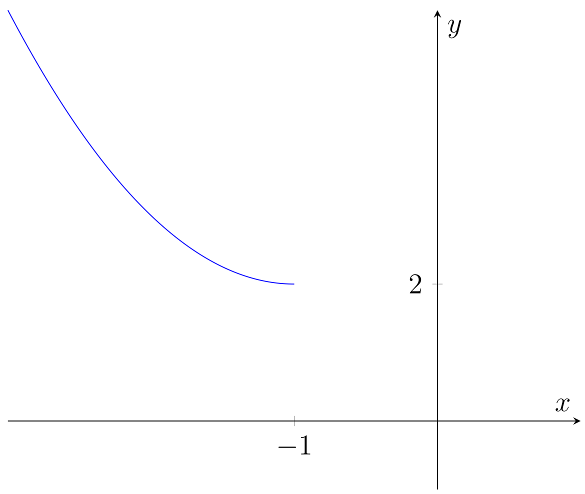
$$f(x) = x^2 + 2x + 3$$

Complete the square for $f(x)$**,**

$$f(x) = (x + 1)^2 - 1 + 3$$

$$f(x) = (x + 1)^2 + 2$$

Sketch the graph of $y = f(x)$,



From the sketch we can tell that the range of f is,

$$y \geq 2$$

Therefore, the final answer is,

$$f(x) = (x+1)^2 + 2, \text{ the range of } f \text{ is } y \geq 2$$

(b) Find an expression of $f^{-1}(x)$.

$$f(x) = (x+1)^2 + 2$$

Start by substituting $f(x)$ with y,

$$y = (x+1)^2 + 2$$

Subtract 2 from both sides,

$$y - 2 = (x+1)^2$$

Take the square root of both sides,

$$\pm\sqrt{y-2} = x + 1$$

Subtract 1 from both sides,

$$x = -1 \pm \sqrt{y-2}$$

Since the domain of f has a $<$ sign, we will take the negative sign,

$$x = -1 - \sqrt{y-2}$$

Therefore, the final answer is,

$$f^{-1}(x) = -1 - \sqrt{x-2}$$

(c) Solve the equation $gf(x) = 13$.

$$f(x) = x^2 + 2x + 3 \quad g(x) = 2x + 1$$

Let's find $gf(x)$ first. Substitute x in $g(x)$ with $f(x)$,

$$g(x) = 2x + 1$$

$$g(f(x)) = 2\left(x^2 + 2x + 3\right) + 1$$

Expand the brackets and simplify,

$$gf(x) = 2x^2 + 4x + 6 + 1$$

$$gf(x) = 2x^2 + 4x + 7$$

Equate gf to 13,

$$2x^2 + 4x + 7 = 13$$

Put all the terms on one side and simplify,

$$2x^2 + 4x + 7 - 13 = 0$$

$$2x^2 + 4x - 6 = 0$$

$$x^2 + 2x - 3 = 0$$

Solve the quadratic,

$$(x + 3)(x - 1) = 0$$

$$x = -3, \quad x = 1$$

Therefore, the final answer is,

$$x = -3, 1$$

15. (a) The curve with equation $y = x^2 + 2x - 5$ is translated by $\binom{-1}{3}$. (9709/13/M/J/22 number 4) Find the equation of the translated curve, giving your answer in the form $y = ax^2 + bx + c$.

$$y = x^2 + 2x - 5$$

Let's start by completing the square,

$$y = (x + 1)^2 - 1 - 5$$

$$y = (x + 1)^2 - 6$$

Let's add a translation of -1 unit in the x-direction,

$$y = (x + 1 + 1)^2 - 6$$

$$y = (x + 2)^2 - 6$$

Let's add a translation of 3 units in the y-direction,

$$y = (x + 2)^2 - 6 + 3$$

$$y = (x + 2)^2 - 3$$

Now let's expand the equation,

$$y = x^2 + 4x + 4 - 3$$

$$y = x^2 + 4x + 1$$

Therefore, the equation of the translated curve is,

$$y = x^2 + 4x + 1$$

(b) The curve with equation $y = x^2 + 2x - 5$ is transformed to a curve with equation $y = 4x^2 + 4x - 5$. Describe fully the single transformation that has been applied.

$$y = x^2 + 2x - 5 \quad y = 4x^2 + 4x - 5$$

Let's define the original curve as $f(x)$,

$$f(x) = x^2 + 2x - 5$$

The transformed curve has been stretched since the coefficient of x^2 has changed. Let's compare the two equations,

$$f(x) = x^2 + 2x - 5 \quad y = 4x^2 + 4x - 5$$

The difference is that x has been replaced with $2x$,

$$f(x) = x^2 + 2x - 5$$

$$f(2x) = (2x)^2 + 2(2x) - 5$$
$$f(2x) = 4x^2 + 4x - 5$$

This means that, to get to the transformed curve, there is a stretch in the x-direction by a scale factor of $\frac{1}{2}$. Therefore, the final answer is,

Stretch in the x-direction, by a scale factor of $\dfrac{1}{2}$