Pure Maths 1

1.5 Trigonometry - Easy

GCEALevelMaths9709

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**AS Level**
Component:	**Pure Mathematics 1**
Topic:	**1.5 Trigonometry**
Difficulty:	**Easy**

Questions

1. Solve the equation $4\sin\theta + \tan\theta = 0$ for $0° < \theta < 180°$. (9709/11/M/J/23 number 1)

2. A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \le x \le 4\pi$. (9709/11/M/J/23 number 7)

   (a) State the greatest and least values of y.

   (b) Sketch the curve.

   (c) State the number of solutions of the equation

   $$2 + 3\sin\frac{1}{2}x = 5 - 2x$$

   for $0 \le x \le 4\pi$.

3. (a) i. By first expanding $(\cos\theta + \sin\theta)^2$, find the three solutions of the equation

   $$(\cos\theta + \sin\theta)^2 = 1$$

   for $0 \le \theta \le \pi$. (9709/12/M/J/23 number 7)

   ii. Hence verify that the only solutions of the equation $\cos\theta + \sin\theta = 1$ for $0 \le \theta \le \pi$ are 0 and $\frac{1}{2}\pi$.

   (b) Prove the identity $\frac{\sin\theta}{\cos\theta+\sin\theta} + \frac{1-\cos\theta}{\cos\theta-\sin\theta} \equiv \frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$.

   (c) Using the results of (a)ii and (b), solve the equation

   $$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1-\cos\theta}{\cos\theta - \sin\theta} = 2(\cos\theta + \sin\theta - 1)$$

   for $0 \le \theta \le \pi$.

4. (a) Show that the equation

   $$3\tan^2 x - 3\sin^2 x - 4 = 0$$

   may be expressed in the form $a\cos^4 x + b\cos^2 x + c = 0$, where a, b and c are constants to be found. (9709/13/M/J/23 number 4)

   (b) Hence solve the equation $3\tan^2 x - 3\sin^2 x - 4 = 0$ for $0° \le x \le 180°$.

5. It is given that the solution to the equation $6\sqrt{y} + \frac{2}{\sqrt{y}} - 7 = 0$ is $y = \frac{1}{4}, \frac{4}{9}$. Hence solve the equation $6\sqrt{\tan x} + \frac{2}{\sqrt{\tan x}} - 7 = 0$ for $0° \le x \le 360°$. (9709/13/M/J/22 number 5b)

6. (a) Show that the equation

   $$\frac{1}{\sin\theta + \cos\theta} + \frac{1}{\sin\theta - \cos\theta} = 1$$

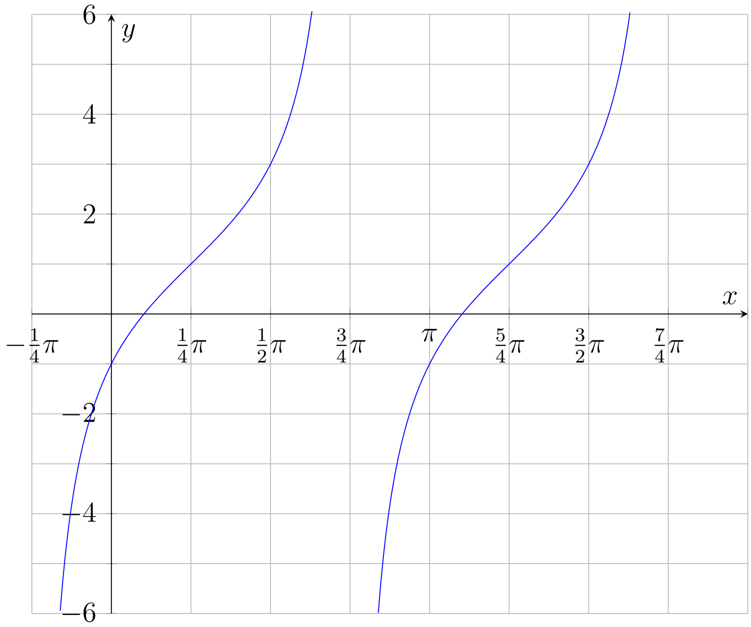
   may be expressed in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. (9709/11/O/N/22 number 6)

   (b) Hence solve the equation $\frac{1}{\sin\theta+\cos\theta} + \frac{1}{\sin\theta-\cos\theta} = 1$ for $0° \le \theta \le 360°$.

7. Solve the equation $8\cos^2\theta - 10\cos\theta + 2 = 0$ for $0° \le \theta \le 180°$. (9709/12/O/N/22 number 3b)

8. Solve the equation $8\sin^2\theta + 6\cos\theta + 1 = 0$ for $0° < \theta < 180°$. (9709/13/O/N/22 number 1)

9.



The diagram shows part of the graph of $y = a\tan(x - b) + c$. Given that $0 < b < \pi$, state the values of the constants a, b and c. (9709/11/M/J/21 number 4)

10. (a) Prove the identity $\frac{1-2\sin^2\theta}{1-\sin^2\theta} \equiv 1 - \tan^2\theta$. (9709/11/M/J/21 number 7)

    (b) Hence solve the equation $\frac{1-2\sin^2\theta}{1-\sin^2\theta} = 2\tan^4\theta$ for $0° \leq \theta \leq 180°$.

11. (a) Show that the equation

$$\frac{\tan x + \sin x}{\tan x - \sin x} = k,$$

where k is a constant, may be expressed as

$$\frac{1 + \cos x}{1 - \cos x} = k$$

. (9709/13/M/J/21 number 4)

    (b) Hence express $\cos x$ in terms of k.

    (c) Hence solve the equation $\frac{\tan x+\sin x}{\tan x-\sin x} = 4$ for $-\pi < x < \pi$.

Answers

1. Solve the equation $4\sin\theta + \tan\theta = 0$ for $0° < \theta < 180°$. (9709/11/M/J/23 number 1)

$$4\sin\theta + \tan\theta = 0$$

Let's rewrite $\tan\theta$ using the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$,

$$4\sin\theta + \frac{\sin\theta}{\cos\theta} = 0$$

Multiply through by $\cos\theta$ to get rid of the denominator,

$$4\sin\theta\cos\theta + \sin\theta = 0$$

Factor out $\tan\theta$,

$$\sin\theta\,(4\cos\theta + 1) = 0$$

Equate the two trig expressions to 0 and simplify,

$$\sin\theta = 0 \quad 4\cos\theta + 1 = 0$$

$$\sin\theta = 0 \quad 4\cos\theta = -1$$

$$\sin\theta = 0 \quad \cos\theta = \frac{-1}{4}$$

Make θ the subject of the formula and solve the trig equations,

$$\theta = \sin^{-1}(0) \quad \theta = \cos^{-1}\left(\frac{-1}{4}\right)$$

$$P.V = 0 \quad P.V = 104.4775122$$

$$P.V(-1)^n + 180n \quad \pm P.V + 360n$$

At $n = 1$

$$P.V(-1)^1 + 180(1) = 180 \quad \pm P.V + 360(1) = 255.5225, 464.4775$$

Select only the answers that are within the given interval,

$$\theta = 104.4775122$$

Therefore, the final answer is,

$$\theta = 104.5°$$

2. A curve has equation $y = 2 + 3\sin\frac{1}{2}x$ for $0 \leq x \leq 4\pi$. (9709/11/M/J/23 number 7)

   (a) State the greatest and least values of y.

$$y = 2 + 3\sin\frac{1}{2}x$$

To get the least value of y substitute $\sin\frac{1}{2}x$ with -1,

$$y = 2 + 3(-1)$$

$$y = -1$$

The least value is,

$$y = -1$$

To get the greatest value of y substitute $\sin\frac{1}{2}x$ with 1,

$$y = 2 + 3(1)$$

$$y = 5$$

The greatest value is,

$$y = 5$$

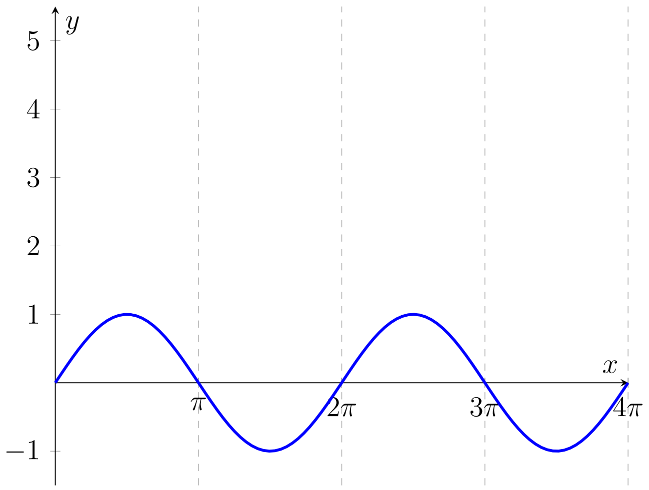
Therefore, the final answer is,

$$\text{The greatest value is } 5. \text{ The least value is } -1.$$

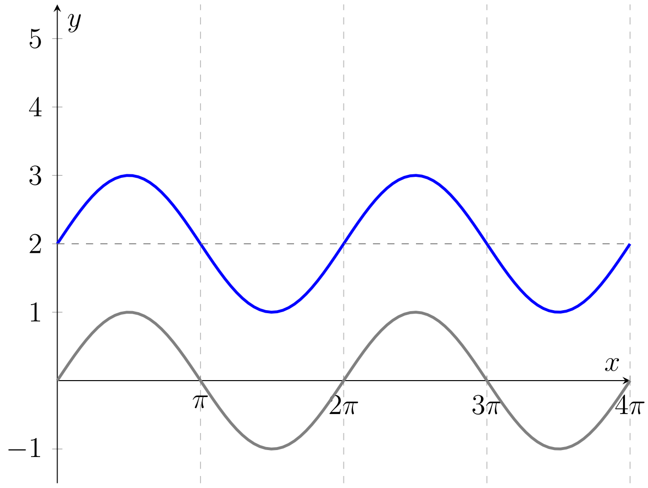
(b) Sketch the curve.

$$y = 2 + 3\sin\frac{1}{2}x \text{ for } 0 \leq x \leq 4\pi$$

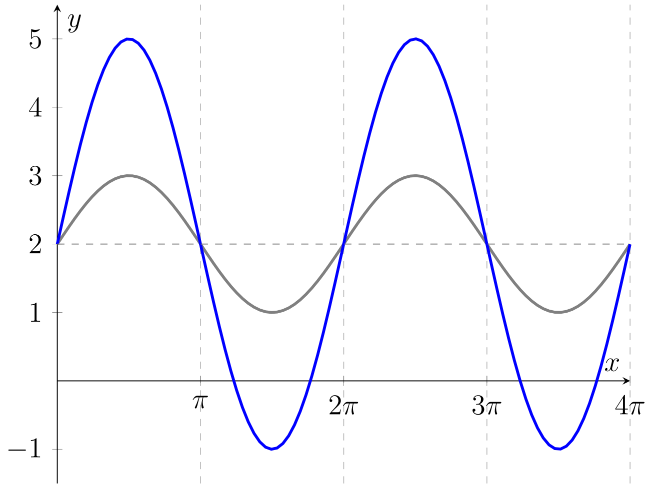
We will sketch the curve using transformations. Start by sketching the graph of $y = \sin x$ for $0 \leq x \leq 4\pi$,



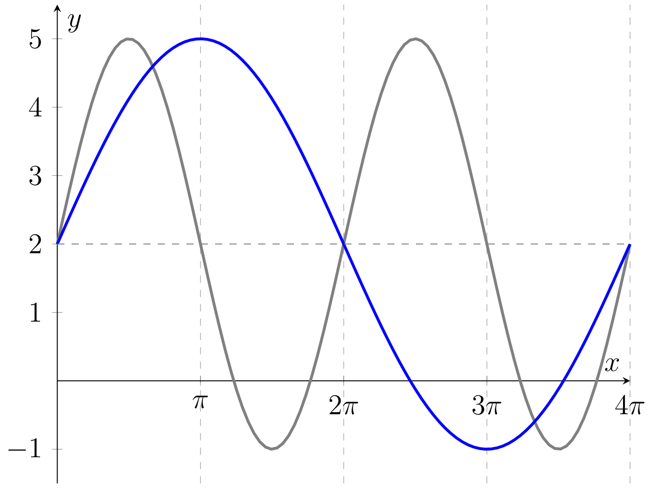
The $\color{blue}{2}$ is a translation in the y-direction,



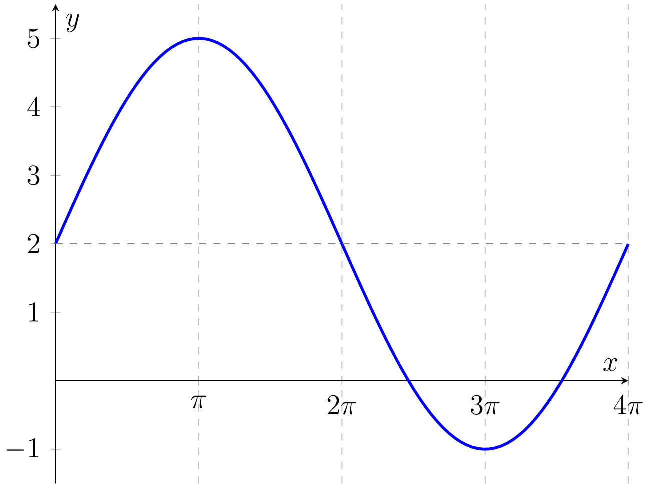
The $\color{green}{3}$ is a stretch in the y-direction,



The $\color{red}{\frac{1}{2}}$ is a stretch in the x-direction. This means the graph has $\frac{1}{2}$ of a complete period from 0 to 2π,



Therefore, the final answer is,

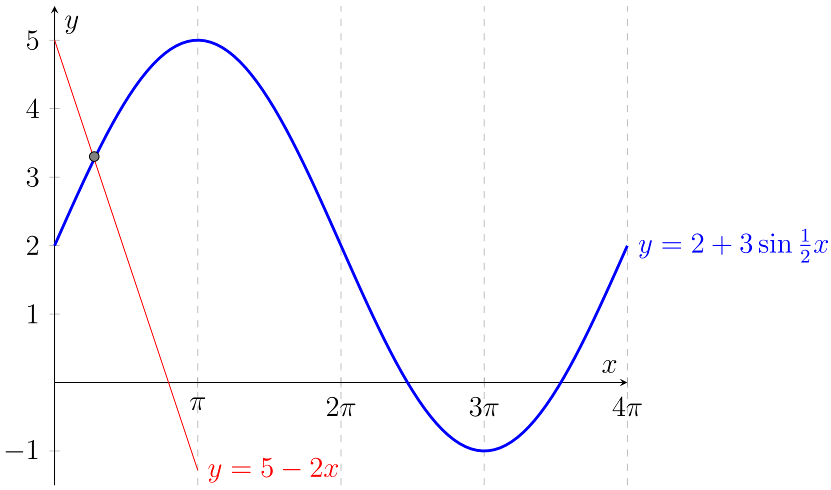


(c) State the number of solutions of the equation

$$2 + 3\sin\frac{1}{2}x = 5 - 2x$$

for $0 \leq x \leq 4\pi$.

Sketch the graph of $y = 5 - 2x$ on the same plane as the graph of $y = 2 + 3\sin\frac{1}{2}x$,



The number of solutions is the number of points of intersection. Therefore, the final answer is,

$$1$$

3. (a) i. By first expanding $(\cos\theta + \sin\theta)^2$, find the three solutions of the equation

$$(\cos\theta + \sin\theta)^2 = 1$$

for $0 \leq \theta \leq \pi$. (9709/12/M/J/23 number 7)

$$(\cos\theta + \sin\theta)^2$$

Let's expand,

$$\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta$$
$$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$$

Use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ **to simplify,**

$$1 + 2\sin\theta\cos\theta$$

$$(\cos\theta + \sin\theta)^2 = 1 + 2\sin\theta\cos\theta$$

Now that we have expanded, the question requires us to solve the equation,

$$(\cos\theta + \sin\theta)^2 = 1$$

$$1 + 2\sin\theta\cos\theta = 1$$
$$2\sin\theta\cos\theta = 0$$

Equate each trig function to 0 **and solve,**

$$2\sin\theta = 0 \quad \cos\theta = 0$$

$$\sin\theta = 0 \quad \cos\theta = 0$$
$$\theta = \sin^{-1}(0) \quad \theta = \cos^{1}(0)$$
$$P.V = 0 \quad P.V = \frac{\pi}{2}$$
$$P.V(-1)^n + \pi n \quad \pm P.V + 2\pi n$$
$$\text{At } n = 1$$
$$P.V(-1)^1 + \pi(1) = \pi \quad \pm P.V + 2\pi(1) = \frac{3}{2}\pi, \frac{5}{2}\pi$$

Select only the answers that are within the given interval,

$$\theta = 0, \frac{\pi}{2}, \pi$$

Therefore, the final answer is,

$$\theta = 0, \frac{\pi}{2}, \pi$$

ii. Hence verify that the only solutions of the equation $\cos\theta + \sin\theta = 1$ for $0 \le \theta \le \pi$ are 0 and $\frac{1}{2}\pi$.

Substitute the solutions from part i into $\cos\theta+\sin\theta=1$ and check if they satisfy the equation,

$$\cos\theta+\sin\theta=1$$

$$\text{At } \theta=0$$

$$\cos 0+\sin 0$$

$$1+0$$

$$1$$

$$\cos 0+\sin 0=1$$

$$\text{At } \theta=\frac{\pi}{2}$$

$$\cos\frac{\pi}{2}+\sin\frac{\pi}{2}$$

$$0+1$$

$$1$$

$$\cos\frac{\pi}{2}+\sin\frac{\pi}{2}=1$$

$$\text{At } \theta=\pi$$

$$\cos\pi+\sin\pi$$

$$-1+0$$

$$-1$$

$$\cos\pi+\sin\pi=-1$$

$$\cos\pi+\sin\pi\neq 1$$

Therefore, we have verified that the only solutions are,

$$\theta=0,\frac{1}{2}\pi$$

(b) Prove the identity $\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}\equiv\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$.

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}\equiv\frac{\cos\theta+\sin\theta-1}{1-2\sin^2\theta}$$

Let's work from the left hand side to the right hand side,

$$\frac{\sin\theta}{\cos\theta+\sin\theta}+\frac{1-\cos\theta}{\cos\theta-\sin\theta}$$

Combine the two terms into one term by finding their common denominator,

$$\frac{\sin\theta(\cos\theta-\sin\theta)+(1-\cos\theta)(\cos\theta+\sin\theta)}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)}$$

Expand both the numerator and the denominator,

$$\frac{\sin\theta\cos\theta - \sin^2\theta + \cos\theta + \sin\theta - \cos^2\theta - \sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

Group like terms and simplify,

$$\frac{\cos\theta + \sin\theta - \sin^2\theta - \cos^2\theta}{\cos^2\theta - \sin^2\theta}$$

Factor out -1 in the last two terms in the numerator,

$$\frac{\cos\theta + \sin\theta - \left(\sin^2\theta + \cos^2\theta\right)}{\cos^2\theta - \sin^2\theta}$$

Let's use the identity $\sin^2\theta + \cos^2\theta \equiv 1$ to simplify the numerator,

$$\frac{\cos\theta + \sin\theta - (1)}{\cos^2\theta - \sin^2\theta}$$

$$\frac{\cos\theta + \sin\theta - 1}{\cos^2\theta - \sin^2\theta}$$

Let's use the identity $\cos^2\theta \equiv 1 - \sin^2\theta$ to simplify the denominator,

$$\frac{\cos\theta + \sin\theta - 1}{1 - \sin^2\theta - \sin^2\theta}$$

Simplify the denominator,

$$\frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta}$$

Therefore, we have proved that,

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} \equiv \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta}$$

(c) Using the results of (a)ii and (b), solve the equation

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} = 2(\cos\theta + \sin\theta - 1)$$

for $0 \leq \theta \leq \pi$.

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} \equiv \frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta}$$

Use the identity we proved in (b) to simplify,

$$\frac{\sin\theta}{\cos\theta + \sin\theta} + \frac{1 - \cos\theta}{\cos\theta - \sin\theta} = 2(\cos\theta + \sin\theta - 1)$$

$$\frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} = 2(\cos\theta + \sin\theta - 1)$$

Put all the terms on one side,

$$\frac{\cos\theta + \sin\theta - 1}{1 - 2\sin^2\theta} - 2(\cos\theta + \sin\theta - 1) = 0$$

Factor out $\cos\theta + \sin\theta - 1$ **since it is common in both terms,**

$$(\cos\theta + \sin\theta - 1)\left(\frac{1}{1 - 2\sin^2\theta} - 2\right) = 0$$

Note: If you divide by $\cos\theta + \sin\theta - 1$ **you lose solutions.**

Equate each bracket to 0**,**

$$\cos\theta + \sin\theta - 1 = 0 \quad \frac{1}{1 - 2\sin^2\theta} - 2 = 0$$

Let's solve the first equation,

$$\cos\theta + \sin\theta - 1 = 0$$

If you move -1 **to the right hand side, the equation is exactly the same as the one in (a) ii.**

$$\cos\theta + \sin\theta = 1$$

In part (a) ii, we were given the solutions of this equation to be,

$$\theta = 0, \frac{\pi}{2}$$

Now let's solve the second equation,

$$\frac{1}{1 - 2\sin^2\theta} - 2 = 0$$

Add 2 **to both sides,**

$$\frac{1}{1 - 2\sin^2\theta} = 2$$

Multiply both sides by the denominator,

$$1 = 2\left(1 - 2\sin^2\theta\right)$$

Expand the bracket on the right hand side,

$$1 = 2 - 4\sin^2\theta$$

Make $\sin^2\theta$ **the subject of the formula,**

$$4\sin^2\theta = 2 - 1$$

$$4\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{4}$$

Take the square root of both sides,

$$\sin\theta = \pm\sqrt{\frac{1}{4}}$$

Solve the two trig equations separately,

$$\sin\theta = -\sqrt{\frac{1}{4}} \quad \sin\theta = \sqrt{\frac{1}{4}}$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{1}{4}}\right) \quad \theta = \sin^{-1}\left(\sqrt{\frac{1}{4}}\right)$$

$$P.V = -\frac{1}{6}\pi \quad P.V = \frac{1}{6}\pi$$

$$P.V(-1)^n + \pi n$$

At $n = 1$

$$P.V(-1)^1 + \pi(1) = \frac{7}{6}\pi \quad P.V(-1)^1 + \pi(1) = \frac{5}{6}\pi$$

Select only the solutions that are within the given interval,

$$\theta = \frac{1}{6}\pi, \frac{5}{6}\pi$$

Remember that from the first equation we got the solutions,

$$\theta = 0, \frac{\pi}{2}$$

Therefore, the final answer is,

$$\theta = 0, \frac{1}{6}\pi, \frac{\pi}{2}, \frac{5}{6}\pi$$

4. (a) Show that the equation

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

may be expressed in the form $a\cos^4 x + b\cos^2 x + c = 0$, where a, b and c are constants to be found. (9709/13/M/J/23 number 4)

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

Use the identity $\tan^2 x \equiv \frac{\sin^2 x}{\cos^2 x}$**,**

$$3\left(\frac{\sin^2 x}{\cos^2 x}\right) - 3\sin^2 x - 4 = 0$$

Multiply through by $\cos^2 x$ **to get rid of the denominator,**

$$3\sin^2 x - 3\sin^2 x\cos^2 x - 4\cos^2 x = 0$$

Since we want an equation in terms of $\cos x$ use the identity $\sin^2 x \equiv 1 - \cos^2 x$ to get rid of $\sin^2 x$,

$$3\left(1-\cos^2 x\right) - 3\left(1-\cos^2 x\right)\cos^2 x - 4cos^2 x = 0$$

Expand the brackets,

$$3 - 3\cos^2 x - 3\cos^2 x + 3\cos^4 x - 4\cos^2 x = 0$$

Group like terms and simplify,

$$3\cos^4 x - 3\cos^2 x - 3\cos^2 x - 4\cos^2 x + 3 = 0$$

$$3\cos^4 x - 10\cos^2 x + 3 = 0$$

Therefore, the final answer is,

$$3\cos^4 x - 10\cos^2 x + 3 = 0$$

(b) Hence solve the equation $3\tan^2 x - 3\sin^2 x - 4 = 0$ for $0° \leq x \leq 180°$.

$$3\tan^2 x - 3\sin^2 x - 4 = 0$$

We know that $3\tan^2 x - 3\sin^2 x - 4$ can be written as,

$$3\cos^4 x - 10\cos^2 x + 3$$

Let's use that to simplify our equation,

$$3\cos^4 x - 10\cos^2 x + 3 = 0$$

$$3\cos^4 x - 10\cos^2 x + 3 = 0$$

Notice that this is a hidden quadratic. Let's rewrite $\cos^4 x$ in terms of $\cos^2 x$,

$$3\left(\cos^2 x\right)^2 - 10\cos^2 x + 3 = 0$$

$$3\left(\cos^2 x\right)^2 - 10\cos^2 x + 3 = 0$$

Solve the quadratic,

$$\left(3\cos^2 x - 1\right)\left(\cos^2 x - 3\right) = 0$$

$$3\cos^2 x - 1 = 0 \quad \cos^2 x - 3 = 0$$

Solve the trig equations,

$$3\cos^2 x = 1 \quad \cos^2 x = 3$$
$$\cos^2 x = \frac{1}{3} \quad \cos^2 x = 3$$
$$\cos x = \pm\sqrt{\frac{1}{3}} \quad \cos x = \pm\sqrt{3}$$
$$\cos x = -\sqrt{\frac{1}{3}} \quad \cos x = \sqrt{\frac{1}{3}} \quad \cos x = \pm\sqrt{3}$$
$$x = \cos^{-1}\left(-\sqrt{\frac{1}{3}}\right) \quad x = \cos^{-1}\left(\sqrt{\frac{1}{3}}\right) \quad x = \text{No Solutions}$$
$$P.V = 125.2644 \quad P.V = 54.7356$$
$$\pm P.V + 360n$$
$$\text{At } n = 1$$
$$\pm P.V + 360(1) = 234.7..., 485.2... \quad \pm P.V + 360(1) = 305.2..., 414.7...$$

Select only the answers that are within the given interval,

$$x = 54.7^\circ, 125.3^\circ$$

Therefore, the final answer is,

$$x = 54.7^\circ, 125.3^\circ$$

5. It is given that the solution to the equation $6\sqrt{y} + \frac{2}{\sqrt{y}} - 7 = 0$ is $y = \frac{1}{4}, \frac{4}{9}$. Hence solve the equation $6\sqrt{\tan x} + \frac{2}{\sqrt{\tan x}} - 7 = 0$ for $0^\circ \leq x \leq 360^\circ$. (9709/13/M/J/22 number 5b)

Let's compare the two equations that we have,

$$6\sqrt{y} + \frac{2}{\sqrt{y}} - 7 = 0 \quad 6\sqrt{\tan x} + \frac{2}{\sqrt{\tan x}} - 7 = 0$$

In the new equation y has been replaced with $\tan x$. Since we know that,

$$y = \frac{1}{4}, \frac{4}{9}$$

We can say that,

$$\tan x = \frac{1}{4}, \frac{4}{9}$$

Let's write them as two separate equations,

$$\tan x = \frac{1}{4} \quad \tan x = \frac{4}{9}$$

Solve the two trig equations,

$$x = \tan^{-1}\left(\frac{1}{4}\right) \quad x = \tan^{-1}\left(\frac{4}{9}\right)$$

$$P.V = 14.0362 \quad P.V = 23.9625$$

$$P.V = 180n$$

At $n = 1$

$$P.V + 180(1) = 194.0362 \quad P.V + 180(1) = 203.9625$$

At $n = 2$

$$P.V + 180(2) = 374.0362 \quad P.V + 180(2) = 383.9625$$

Select only the answers that are within the given interval,

$$x = 14.0^\circ, 24.0^\circ, 194.0^\circ, 204.0^\circ$$

Therefore, the final answer is,

$$x = 14.0^\circ, 24.0^\circ, 194.0^\circ, 204.0^\circ$$

6. (a) Show that the equation

$$\frac{1}{\sin\theta + \cos\theta} + \frac{1}{\sin\theta - \cos\theta} = 1$$

may be expressed in the form $a\sin^2\theta + b\sin\theta + c = 0$, where a, b and c are constants to be found. (9709/11/O/N/22 number 6)

$$\frac{1}{\sin\theta + \cos\theta} + \frac{1}{\sin\theta - \cos\theta} = 1$$

Multiply through by $(\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$ to get rid of the denominators,

$$\sin\theta - \cos\theta + \sin\theta + \cos\theta = (\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$$

Simplify the left hand side,

$$2\sin\theta = (\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$$

Expand the brackets on the right hand side,

$$2\sin\theta = \sin^2\theta - \cos^2\theta$$

We want everything to be in terms of $\sin\theta$. Use the identity $\cos^2\theta \equiv 1 - \sin^2\theta$ to get rid of $\cos^2\theta$,

$$2\sin\theta = \sin^2\theta - \left(1 - \sin^2\theta\right)$$

Expand the bracket and simplify,

$$2\sin\theta = \sin^2\theta - 1 + \sin^2\theta$$

$$2\sin\theta = 2\sin^\theta - 1$$

Put all the terms on one side,

$$2\sin^2\theta - 2\sin\theta - 1 = 0$$

Therefore, the final answer is,

$$2\sin^2\theta - 2\sin\theta - 1 = 0$$

(b) Hence solve the equation $\frac{1}{\sin\theta+\cos\theta} + \frac{1}{\sin\theta-\cos\theta} = 1$ for $0° \leq \theta \leq 360°$.

$$\frac{1}{\sin\theta + \cos\theta} + \frac{1}{\sin\theta - \cos\theta} = 1$$

We can replace $\frac{1}{\sin\theta+\cos\theta} + \frac{1}{\sin\theta-\cos\theta}$ **with** $2\sin^2\theta - 2\sin\theta - 1$**,**

$$2\sin^2\theta - 2\sin\theta - 1 = 0$$

Let's use the quadratic formula to solve the quadratic equation,

$$\sin\theta = \frac{2 \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$\sin\theta = \frac{1 \pm \sqrt{3}}{2}$$

Solve the two trig equations separately,

$$\sin\theta = \frac{1-\sqrt{3}}{2} \quad \sin\theta = \frac{1+\sqrt{3}}{2}$$

$$\theta = \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right) \quad \theta = \sin^{-1}\left(\frac{1+\sqrt{3}}{2}\right)$$

$$P.V = -21.4707 \quad \theta = \text{No Solutions}$$

$$P.V(-1)^n + 180n$$

$$\text{At } n = 1$$

$$P.V(-1)^1 + 180(1) = 201.4707$$

$$\text{At } n = 2$$

$$P.V(-1)^2 + 180(2) = 338.5293$$

$$\text{At } n = 3$$

$$P.V(-1)^3 + 180(3) = 561.4707$$

Select only the answers that are within the given interval,

$$\theta = 201.5°, 338.5°$$

Therefore, the final answer is,

$$\theta = 201.5°, 338.5°$$

7. Solve the equation $8\cos^2\theta - 10\cos\theta + 2 = 0$ for $0° \leq \theta \leq 180°$. (9709/12/O/N/22 number 3b)

$$8\cos^2\theta - 10\cos\theta + 2 = 0$$

Solve the quadratic equation,

$$(4\cos\theta - 1)(\cos\theta - 1) = 0$$

$$4\cos\theta - 1 = 0 \quad \cos\theta - 1 = 0$$

Solve the trig equations,

$$4\cos\theta = 1 \quad \cos\theta = 1$$

$$\cos\theta = \frac{1}{4} \quad \cos\theta = 1$$

$$\theta = \cos^{-1}\left(\frac{1}{4}\right) \quad \theta = \cos^{-1}(1)$$

$$P.V = 75.5225 \quad P.V = 0$$

$$\pm P.V + 360n$$

$$\text{At } n = 1$$

$$\pm P.V + 360(1) = 284.4775, 435.5225 \quad \pm P.V + 360(1) = 360$$

Select only the answers that are within the given interval,

$$\theta = 0°, 75.5°$$

Therefore, the final answer is,

$$\theta = 0°, 75.5°$$

8. Solve the equation $8\sin^2\theta + 6\cos\theta + 1 = 0$ for $0° < \theta < 180°$. (9709/13/O/N/22 number 1)

$$8\sin^2\theta + 6\cos\theta + 1 = 0$$

To be able to solve this equation we have to create a quadratic equation in terms of $\cos\theta$**. Let's use the identity** $\sin^2\theta \equiv 1 - \cos^2\theta$ **to get rid of** $\sin^2\theta$**,**

$$8\left(1 - \cos^2\theta\right) + 6\cos\theta + 1 = 0$$

Expand the brackets,

$$8 - 8\cos^2\theta + 6\cos\theta + 1 = 0$$

Group like terms and simplify,

$$-8\cos^2\theta + 6\cos\theta + 9 = 0$$

$$8\cos^2\theta - 6\cos\theta - 9 = 0$$

Now we have a quadratic equation in terms of $\cos\theta$. Solve the quadratic equation,

$$(4\cos\theta + 3)(2\cos\theta - 3) = 0$$

$$4\cos\theta + 3 = 0 \quad 2\cos\theta - 3 = 0$$

Solve the trig equations,

$$4\cos\theta = -3 \quad 2\cos\theta = 3$$

$$\cos\theta = \frac{-3}{4} \quad \cos\theta = \frac{3}{2}$$

$$\theta = \cos^{-1}\left(\frac{-3}{4}\right) \quad \theta = \cos^{-1}\left(\frac{3}{2}\right)$$

$$\theta = 138.5904 \quad \theta = \text{No Solutions}$$

$$\pm P.V + 360n$$

$$\text{At } n = 1$$

$$\pm P.V + 360(1) = 221.4096, 498.5904$$

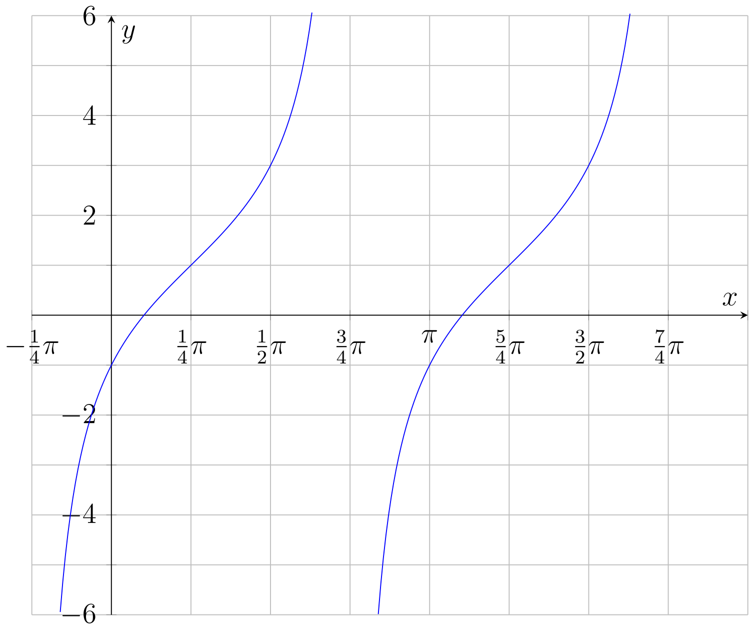
Select only the answers that are within the given interval,

$$\theta = 138.6°$$

Therefore, the final answer is,

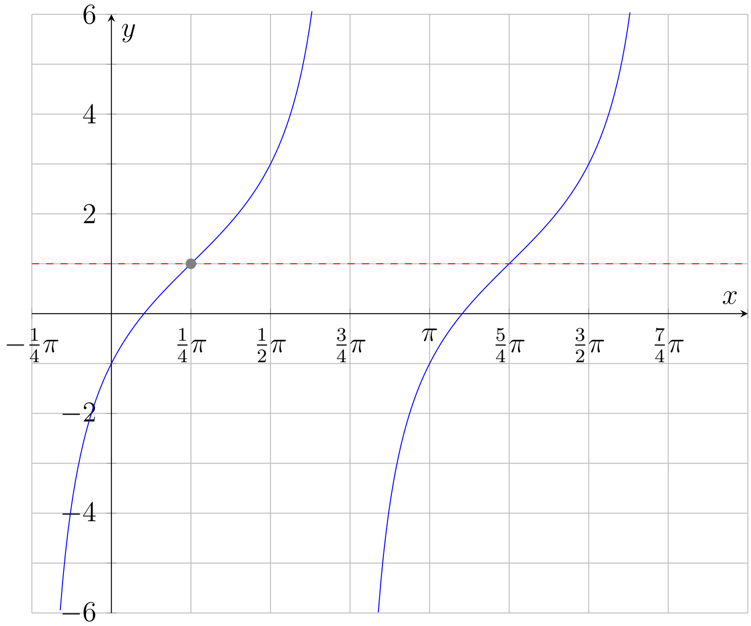
$$\theta = 138.6°$$

9.



The diagram shows part of the graph of $y = a\tan(x - b) + c$. Given that $0 < b < \pi$, state the values of the constants a, b and c. (9709/11/M/J/21 number 4)

On the graph of $y = \tan x$ the centre of the curve lies on $(0, 0)$ but in this case it lies on $\left(\frac{1}{4}\pi, 1\right)$. This means our graph has translated by $\frac{1}{4}\pi$ units in the x-axis, and 1 unit in the y-axis,



$$b = \frac{1}{4}\pi$$
$$c = 1$$

To find a let's pick a point the lies on the curve. Let's use $(0, -1)$,

$$\text{At } (0, -1)$$

$$-1 = a \tan\left(0 - \frac{1}{4}\right) + 1$$

Note: Make sure to substitute in the values of b and c.

$$-1 = a(-1) + 1$$

$$-1 = -a + 1$$

$$a = 1 + 1$$

$$a = 2$$

Therefore, the final answer is,

$$a = 2 \quad b = \frac{1}{4}\pi \quad c = 1$$

10. (a) Prove the identity $\frac{1-2\sin^2\theta}{1-\sin^2\theta} \equiv 1 - \tan^2\theta$. (9709/11/M/J/21 number 7)

$$\frac{1 - 2\sin^2\theta}{1 - \sin^2\theta} \equiv 1 - \tan^2\theta$$

Let's work from the left hand side to the right hand side,

$$\frac{1-2\sin^2\theta}{1-\sin^2\theta}$$

Use the identity $\cos^2\theta = 1-\sin^2\theta$ **to simplify the denominator,**

$$\frac{1-2\sin^2\theta}{\cos^2\theta}$$

$$\frac{1-2\sin^2\theta}{\cos^2\theta}$$

Use the identity $\sin^2\theta+\cos^2\theta \equiv 1$ **to simplify the numerator,**

$$\frac{sin^2\theta+\cos^2\theta-2\sin^2\theta}{\cos^2\theta}$$

$$\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}$$

Distribute the denominator and the split the fraction,

$$\frac{\cos^2\theta}{\cos^2\theta}-\frac{\sin^2\theta}{\cos^2\theta}$$

$$1-\frac{\sin^2\theta}{\cos^2\theta}$$

Use the identity $\tan^2\theta \equiv \frac{\sin^2\theta}{\cos^2\theta}$**,**

$$1-\tan^2\theta$$

$$1-\tan^2\theta$$

Therefore, we have proved that,

$$\frac{1-2\sin^2\theta}{1-\sin^2\theta} \equiv 1-\tan^2\theta$$

(b) Hence solve the equation $\frac{1-2\sin^2\theta}{1-\sin^2\theta} = 2\tan^4\theta$ for $0° \leq \theta \leq 180°$.

$$\frac{1-2\sin^2\theta}{1-\sin^2\theta} = 2\tan^4\theta$$

Substitute $\frac{1-2\sin^2\theta}{1-\sin^2\theta}$ **with** $1-\tan^2\theta$**,**

$$1-\tan^2\theta = 2\tan^4\theta$$

Put all the terms on one side,

$$2\tan^4\theta+\tan^2\theta-1=0$$

Notice that this is a hidden quadratic. Rewrite $\tan^4\theta$ in terms of $\tan^2\theta$,

$$2\left(\tan^2\theta\right)^2 + \tan^2\theta - 1 = 0$$

Solve the quadratic equation,

$$\left(2\tan^{\theta} - 1\right)\left(\tan^2\theta + 1\right) = 0$$

$$2\tan^2\theta - 1 = 0 \quad \tan^2\theta + 1 = 0$$

Solve the trig equations,

$$2\tan^2\theta = 1 \quad \tan^2\theta = -1$$

$$\tan^2\theta = \frac{1}{2} \quad \tan^2\theta = -1$$

$$\tan\theta = \pm\sqrt{\frac{1}{2}} \quad \tan\theta = \sqrt{-1}$$

$$\tan\theta = \pm\sqrt{\frac{1}{2}} \quad \theta = \text{No Solutions}$$

$$\tan\theta = -\sqrt{\frac{1}{2}} \quad \tan\theta = \sqrt{\frac{1}{2}}$$

$$\theta = \tan^{-1}\left(-\sqrt{\frac{1}{2}}\right) \quad \theta = \tan^{-1}\left(\sqrt{\frac{1}{2}}\right)$$

$$P.V = -35.2644 \quad P.V = 35.2644$$

$$P.V + 180n$$

$$\text{At } n = 1$$

$$P.V + 180(1) = 144.7356 \quad P.V + 180(1) = 215.2644$$

$$\text{At } n = 2$$

$$P.V + 180(2) = 324.7356 \quad P.V + 180(2) = 395.2644$$

Select only the values that are within the given interval,

$$\theta = 35.3°, 144.7°$$

Therefore, the final answer is,

$$\theta = 35.3°, 144.7°$$

11. (a) Show that the equation

$$\frac{\tan x + \sin x}{\tan x - \sin x} = k,$$

where k is a constant, may be expressed as

$$\frac{1 + \cos x}{1 - \cos x} = k$$

. (9709/13/M/J/21 number 4)

This means that they want us to show that,

$$\frac{\tan x + \sin x}{\tan x - \sin x} = \frac{1 + \cos x}{1 - \cos x}$$

Let's work from the left hand side to the right hand side,

$$\frac{\tan x + \sin x}{\tan x - \sin x}$$

Use the identity $\tan x = \frac{\sin x}{\cos x}$ **to get rid of** $\tan x$**,**

$$\frac{\frac{\sin x}{\cos x} + \sin x}{\frac{\sin x}{\cos x} - \sin x}$$

Find the common denominator and combine the terms,

$$\frac{\frac{\sin x + \sin x \cos x}{\cos x}}{\frac{\sin x - \sin x \cos x}{\cos x}}$$

This can be written as,

$$\frac{\sin x + \sin x \cos x}{\cos x} \times \frac{\cos x}{\sin x - \sin x \cos x}$$

This simplifies to give,

$$\frac{\sin x + \sin x \cos x}{\sin x - \sin x \cos x}$$

Factor out $\sin x$**,**

$$\frac{\sin x(1 + \cos x)}{\sin x(1 - \cos x)}$$

Cancel out $\sin x$**,**

$$\frac{1 + \cos x}{1 - \cos x}$$

Therefore, we have proved that,

$$\frac{\tan x + \sin x}{\tan x - \sin x} = \frac{1 + \cos x}{1 - \cos x}$$

(b) Hence express $\cos x$ in terms of k.

$$\frac{1 + \cos x}{1 - \cos x} = k$$

Multiply both sides by $1 - \cos x$**,**

$$1 + \cos x = k(1 - \cos x)$$

Expand the bracket on the right hand side,

$$1 + \cos x = k - k \cos x$$

Put all terms containing $\cos x$ on one side,

$$\cos x + k\cos x = k - 1$$

Factor out $\cos x$,

$$\cos x(1 + k) = k - 1$$

Divide both sides by $1 + k$,

$$\cos x = \frac{k-1}{1+k}$$

Therefore, the final answer is,

$$\cos x = \frac{k-1}{1+k}$$

(c) Hence solve the equation $\frac{\tan x + \sin x}{\tan x - \sin x} = 4$ for $-\pi < x < \pi$.

$$\frac{1 + \cos x}{1 - \cos x} = k$$

Notice how in part (b) we reduce the equation,

$$\frac{1 + \cos x}{1 - \cos x} = k \text{ to } \cos x = \frac{k-1}{1+k}$$

This means that,

$$\frac{1 + \cos x}{1 - \cos x} = 4 \text{ to } \cos x = \frac{4-1}{1+4}$$

This gives us,

$$\cos x = \frac{3}{5}$$

Solve the trig equation,

$$x = \cos^{-1}\left(\frac{3}{5}\right)$$

$$P.V = 0.927296$$

$$\pm P.V + 2\pi n$$

$$\text{At } n = 0$$

$$\pm P.V + 2\pi(0) = \pm 0.927296$$

$$\text{At } n = 1$$

$$\pm P.V + 2\pi(1) = 5.35..., 7.21...$$

Select only the answers that are within the given interval,

$$x = \pm 0.927$$

Therefore, the final answer is,

$$x = \pm 0.927$$