Pure Maths 1

1.6 Series - Easy

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**AS Level**
Component:	**Pure Mathematics 1**
Topic:	**1.6 Series**
Difficulty:	**Easy**

Questions

1. The circumference round the trunk of a large tree is measured and found to be 5.00m. After one year the circumference is measured again and found to be 5.02m. (9709/12/F/M/23 number 4)
   (a) Given that the circumferences at yearly intervals form an arithmetic progression, find the circumference 20 years after the first measurement.
   (b) Given instead that the circumferences at yearly intervals form a geometric progression, find the circumference 20 years after the first measurement.

2. (a) Find the first three terms in the expansion, in ascending powers of x, of $(2+3x)^4$. (9709/11/M/J/23 number 2)
   (b) Find the first three terms in the expansion, in ascending powers of x, of $(1-2x)^5$.
   (c) Hence find the coefficient of x^2 in the expansion of $(2+3x)^4(1-2x)^5$.

3. The first three terms of an arithmetic progression are $\frac{p^2}{6}$, $2p-6$ and p. (9709/11/M/J/23 number 6)
   (a) Given that the common difference of the progression is not zero, find the value of p.
   (b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p-6$.

4. The second term of a geometric progression is 16 and the sum to infinity is 100. (9709/12/M/J/23 number 9)
   (a) Find the two possible values of the first term.
   (b) Show that the n-th term of one of the two possible geometric progressions is equal to 4^{n-2} multiplied by the n-th term of the other geometric progression.

5. (a) Give the complete expansion of $\left(x+\frac{2}{x}\right)^5$. (9709/13/M/J/23 number 3)
   (b) In the expansion of $(a+bx^2)\left(x+\frac{2}{x}\right)^5$, the coefficient of x is zero and the coefficient of $\frac{1}{x}$ is 80. Find the values of a and b.

6. The coefficient of x^4 in the expansion of $\left(2x^2+\frac{k^2}{x}\right)^5$ is a. The coefficient of x^2 in the expansion of $(2kx-1)^4$ is b. (9709/11/M/J/22 number 3)
   (a) Find a and b in terms of the constant k.
   (b) Given that $a+b=216$, find the possible values of k.

7. The second and third terms of a geometric progression are 10 and 8 respectively. Find the sum to infinity. (9709/12/M/J/22 number 2)

8. The first, second and third terms of an arithmetic progression are k, $6k$, $k+6$ respectively. (9709/12/M/J/22 number 4)
   (a) Find the value of the constant k.
   (b) Find the sum of the first 30 terms of the progression.

9. The coefficient of x^3 in the expansion of $\left(p+\frac{1}{p}x\right)^4$ is 144. Find the possible values of the constant p. (9709/13/M/J/22 number 1)

10. An arithmetic progression has first term 4 and common difference d. The sum of the first n terms of the progression 5863. (9709/13/M/J/22 number 3)

    (a) Show that $(n-1)d = \frac{11726}{n} - 8$.

    (b) Given that the n-th term is 139, find the values of n and d, giving the value of d as a fraction.

11. A tool for putting fence posts into the ground is called a 'post-rammer'. The distances in millimetres that the post sinks into the ground on each impact of the post-rammer follow a geometric progression. The first three impacts cause the post to sink into the ground by 50mm, 40mm and 32mm respectively. (9709/11/O/N/22 number 7)

    (a) Verify that the 9-th impact is the first time in which the post sinks less than 10mm into the ground.

    (b) Find, to the nearest millimetre, the total depth of the post in the ground after 20 impacts.

    (c) Find the greatest total depth in the ground which could be theoretically achieved.

12. A geometric progression is such that the third term is 1764 and the sum of the second and third terms is 3444. Find the 50-th term. (9709/12/O/N/22 number 4)

13. (a) Find the first three terms in ascending powers of x of the expansion of $(1+2x)^5$. (9709/13/O/N/22 number 3)

    (b) Find the first three terms in ascending powers of x of the expansion of $(1-3x)^4$.

    (c) Hence find the coefficient of x^2 in the expansion of $(1+2x)^5(1-3x)^4$.

14. (a) Find the first three terms in the expansion, in ascending powers of x, of $(1+x)^5$. (9709/12/F/M/21 number 1)

    (b) Find the first three terms in the expansion, in ascending powers of x, of $(1-2x)^6$.

    (c) Hence find the coefficient of x^2 in the expansion of $(1+x)^5(1-2x)^6$.

15. The first term of a geometric progression is $\cos\theta$, where $0 < \theta < \frac{1}{2}\pi$. (9709/12/F/M/21 number 9)

    (a) For the case where the progression is geometric, the sum to infinity is $\frac{1}{\cos\theta}$.

        i. Show that the second term is $\cos\theta\sin^2\theta$.

        ii. Find the sum of the first 12 terms where $\theta = \frac{1}{3}\pi$, giving your answer correct to 4 significant figures.

    (b) For the case where the progression is arithmetic, the first two terms are again $\cos\theta$ and $\cos\theta\sin^2\theta$ respectively. Find the 85-th term when $\theta = \frac{1}{3}\pi$.

16. (a) Find the first three terms in the expansion of $(3-2x)^5$ in ascending powers of x. (9709/11/M/J/21 number 3)

    (b) Hence find the coefficient of x^2 in the expansion of $(4+x)^2(3-2x)^5$.

17. The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively. Given that k is negative, find the sum to infinity of the progression. (9709/11/M/J/21 number 5)

18. (a) Expand $\left(1-\frac{1}{2x}\right)^2$. (9709/11/O/N/21 number 1)

    (b) Find the first four terms in the expansion, in ascending powers of x, of $(1+2x)^6$.

    (c) Hence find the coefficient of x in the expansion of $\left(1-\frac{1}{2x}\right)^2(1+2x)^6$.

Answers

1. The circumference round the trunk of a large tree is measured and found to be 5.00m. After one year the circumference is measured again and found to be 5.02m. (9709/12/F/M/23 number 4)

   (a) Given that the circumferences at yearly intervals form an arithmetic progression, find the circumference 20 years after the first measurement.

   $$a = \text{1st term} = 5 \quad \text{2nd term} = 5.02$$

   Since this is an arithmetic progression, we need to find the common difference,

   $$d = 5.02 - 5$$
   $$d = 0.02$$

   The question requires us to find the circumference after 20 years. This means we need to find the 20-th term of the progression,

   $$u_n = a + (n-1)d$$
   $$u_{20} = 5 + (20-1)(0.02)$$
   $$u_{20} = 5.38$$

   Therefore, the final answer is,

   $$u_{20} = 5.38$$

   (b) Given instead that the circumferences at yearly intervals form a geometric progression, find the circumference 20 years after the first measurement.

   $$a = \text{1st term} = 5 \quad \text{2nd term} = 5.02$$

   Since this is a geometric progression, we need to find the common difference,

   $$r = \frac{5.02}{5}$$
   $$r = 1.004$$

   The question requires us to find the circumference after 20 years. This means we need to find the 20-th term of the progression,

   $$u_n = ar^{n-1}$$
   $$u_{20} = 5(1.004)^{20-1}$$
   $$u_{20} = 5.39$$

   Therefore, the final answer is,

   $$u_{20} = 5.39$$

2. (a) Find the first three terms in the expansion, in ascending powers of x, of $(2+3x)^4$. (9709/11/M/J/23 number 2)

   $$a = 2 \quad b = 3x \quad n = 4$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$2^4 + \binom{4}{1}(2)^{4-1}(3x) + \binom{4}{2}(2)^{4-2}(3x)^2$$

Simplify each term,

$$16 + 96x + 216x^2$$

Therefore, the final answer is,

$$16 + 96x + 216x^2$$

(b) Find the first three terms in the expansion, in ascending powers of x, of $(1 - 2x)^5$.

$$a = 1 \quad b = -2x \quad n = 5$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^5 + \binom{5}{1}(1)^{5-1}(-2x) + \binom{5}{2}(1)^{5-2}(-2x)^2$$

Simplify each term,

$$1 - 10x + 40x^2$$

Therefore, the final answer is,

$$1 - 10x + 40x^2$$

(c) Hence find the coefficient of x^2 in the expansion of $(2 + 3x)^4(1 - 2x)^5$.

$$(2 + 3x)^4 = 16 + 96x + 216x^2 \quad (1 - 2x)^5 = 1 - 10x + 40x^2$$

We are given,

$$(2 + 3x)^4(1 - 2x)^5$$

Let's use the expansions we deduced in (a) and (b),

$$\left(16 + 96x + 216x^2\right)\left(1 - 10x + 40x^2\right)$$

Identify the terms that will multiply to give x^2,

$$16\left(40x^2\right) + 96x\left(-10x\right) + 216x^2(1)$$

Simplify each term,

$$640x^2 - 960x^2 + 216x^2$$

Sum up all the terms,

$$-104x^2$$

The coefficient of x^2 is,

$$-104$$

Therefore, the final answer is,

$$-104$$

3. The first three terms of an arithmetic progression are $\frac{p^2}{6}$, $2p - 6$ and p. (9709/11/M/J/23 number 6)

   (a) Given that the common difference of the progression is not zero, find the value of p.

$$\text{1st term} = \frac{p^2}{6} \quad \text{2nd term} = 2p - 6 \quad \text{3rd term} = p$$

We can find the common difference in two different ways,

$$d = \text{2nd term} - \text{1st term} \quad d = \text{3rd term} - \text{2nd term}$$

Substitute and simplify,

$$d = 2p - 6 - \frac{p^2}{6} \quad d = p - (2p - 6)$$

$$d = 2p - 6 - \frac{p^2}{6} \quad d = p - 2p + 6$$

$$d = 2p - 6 - \frac{p^2}{6} \quad d = 6 - p$$

Equate the two equations together and solve simultaneously,

$$2p - 6 - \frac{p^2}{6} = 6 - p$$

Put all the terms on one side and simplify,

$$\frac{p^2}{6} - p - 2p + 6 + 6 = 0$$

$$\frac{p^2}{6} - 3p + 12 = 0$$

$$p^2 - 18p + 72 = 0$$

Solve the quadratic equation,

$$(p - 6)(p - 12) = 0$$

$$p = 6, \ p = 12$$

Evaluate d,

$$d = 6 - p$$

$$\text{At } p = 6 \quad \text{At } p = 12$$

$$d = 6 - 6 \quad d = 6 - 12$$

$$d = 0 \quad d = -6$$

The question tells that the common difference, d, is not 0, so disregard $p = 6$,

$$p = 12$$

Therefore, the final answer is,

$$p = 12$$

(b) Using this value, find the sum to infinity of the geometric progression with first two terms $\frac{p^2}{6}$ and $2p - 6$.

$$p = 12 \quad a = \text{1st term} = \frac{p^2}{6} \quad \text{2nd term} = 2p - 6$$

Let's evaluate substitute p into the first and second terms,,

$$a = \frac{12^2}{6} \quad \text{2nd term} = 2(12) - 6$$

$$a = 24 \quad \text{2nd term} = 18$$

Let's find the common ratio of this geometric progression,

$$r = \frac{18}{24}$$

$$r = \frac{3}{4}$$

Now let's find the sum to infinity of the geometric progression,

$$S_\infty = \frac{a}{1 - r}$$

$$S_\infty = \frac{24}{1 - \frac{3}{4}}$$

$$S_\infty = 96$$

Therefore, the final answer is,

$$S_\infty = 96$$

4. The second term of a geometric progression is 16 and the sum to infinity is 100. (9709/12/M/J/23 number 9)

(a) Find the two possible values of the first term.

$$\text{2nd term} = 16 \quad S_\infty = 100$$

We can form two equations and solve them simultaneously,

$$ar = 16 \quad \frac{a}{1-r} = 100$$

Make r the subject of the formula in the first equation,

$$r = \frac{16}{a} \quad \frac{a}{1-r} = 100$$

Substitute $r = \frac{16}{a}$ into the second equation,

$$\frac{a}{1-\frac{16}{a}} = 100$$

Multiply through by $1 - \frac{16}{a}$ to get rid of the denominator,

$$a = 100\left(1 - \frac{16}{a}\right)$$

Expand the bracket on the right hand side,

$$a = 100 - \frac{1600}{a}$$

Multiply through by a to get rid of the denominator,

$$a^2 = 100a - 1600$$

Put all the terms on one side,

$$a^2 - 100a + 1600 = 0$$

Solve the quadratic equation,

$$(a-20)(a-80) = 0$$
$$a = 20, 80$$

Therefore, the final answer is,

$$a = 20, 80$$

(b) Show that the n-th term of one of the two possible geometric progressions is equal to 4^{n-2} multiplied by the n-th term of the other geometric progression.

$$a = 20, 80 \quad r = \frac{16}{a}$$

Let's evaluate the possible values of the common ratio,

$$r = \frac{16}{a}$$

$$\text{At } a = 20 \quad \text{At } a = 80$$
$$r = \frac{16}{20} \quad r = \frac{16}{80}$$
$$r = \frac{4}{5} \quad r = \frac{1}{5}$$

The formula for n-th term for a geometric progression is,

$$u_n = ar^{n-1}$$

$$\text{At } a = 20,\ r = \frac{4}{5} \quad \text{At } a = 80,\ r = \frac{1}{5}$$

$$u_n = 20\left(\frac{4}{5}\right)^{n-1} \quad u_n = 80\left(\frac{1}{5}\right)^{n-1}$$

$$u_n = 20\left(\frac{4}{5}\right)^{n-1} \quad u_n = 80\left(\frac{1}{5}\right)^{n-1}$$

The question requires to prove that,

$$n\text{-th term } 1 = n\text{-th term}2 \times 4^{n-2}$$

We can rewrite this to,

$$\frac{n\text{-th term}1}{n\text{-th term}2} = 4^{n-2}$$

Let's divide the two n-th terms,

$$\frac{20\left(\frac{4}{5}\right)^{n-1}}{80\left(\frac{1}{5}\right)^{n-1}}$$

Let's change the fractions to decimals to make it clearer,

$$\frac{20 \times (0.8)^{n-1}}{80 \times (0.2)^{n-1}}$$

This simplifies to give,

$$\frac{1}{4} \times 4^{n-1}$$

We can rewrite this as,

$$4^{-1} \times 4^{n-1}$$

Using laws of indices, add the powers together,

$$4^{-1+n-1}$$

$$4^{n-2}$$

Therefore, we have proved that,

$$n\text{-th term } 1 = n\text{-th term}2 \times 4^{n-2}$$

5. (a) Give the complete expansion of $\left(x + \frac{2}{x}\right)^5$. (9709/13/M/J/23 number 3)

$$a = x \quad b = \frac{2}{x} \quad n = 5$$

Use the formula for binomial expansion to expand,

$$a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$x^5 + {}^5C_1(x)^{5-1}(\frac{2}{x}) + {}^5C_2(x)^{5-2}(\frac{2}{x})^2 +^5 C_3(x)^{5-3}(\frac{2}{x})^3 + {}^5C_4(x)^{5-4}(\frac{2}{x}) + \left(\frac{2}{x}\right)^5$$

Simplify each term,

$$x^5 + 5x^4\left(\frac{2}{x}\right) + 10x^3\left(\frac{2}{x}\right)^2 + 10x^2\left(\frac{2}{x}\right)^3 + 5x\left(\frac{2}{x}\right)^4 + \frac{32}{x^5}$$

$$x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$$

Therefore, the final answer is,

$$x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$$

(b) In the expansion of $(a + bx^2)\left(x + \frac{2}{x}\right)^5$, the coefficient of x is zero and the coefficient of $\frac{1}{x}$ is 80. Find the values of a and b.

$$(a + bx^2)\left(x + \frac{2}{x}\right)^5$$

Replace $\left(x + \frac{2}{x}\right)^5$ with its expansion,

$$(a + bx^2)\, x^5 + 10x^3 + 40x + \frac{80}{x} + \frac{80}{x^3} + \frac{32}{x^5}$$

Identify the terms that will give a term in x when multiplied,

$$a(40x) + bx^2\left(\frac{80}{x}\right)$$

$$40ax + 80bx$$

The coefficient of x is,

$$40a + 80b$$

Equate it to zero,

$$40a + 80b = 0$$

Identify the terms that will give a term in $\frac{1}{x}$ when multiplied,

$$a\left(\frac{80}{x}\right) + bx^2\left(\frac{80}{x^3}\right)$$

$$80a\left(\frac{1}{x}\right) + 80b\left(\frac{1}{x}\right)$$

The coefficient of $\frac{1}{x}$ is,

$$80a + 80b$$

Equate it to 80,

$$80a + 80b = 80$$

We now have two equations in terms of a and b,

$$40a + 80b = 0 \quad 80a + 80b = 80$$

Let's solve them simultaneously. Make a the subject of the formula in the first equation,

$$40a = -80b$$

$$a = -2b$$

Substitute into the second equation and solve for b,

$$80a + 80b = 80$$

$$80(-2b) + 80b = 80$$

$$-160b + 80b = 80$$

$$-80b = 80$$

$$b = -1$$

Evaluate a,

$$a = -2b$$

$$a = -2(-1)$$

$$a = 2$$

Therefore, the final answer is,

$$a = 2 \quad b = -1$$

6. The coefficient of x^4 in the expansion of $\left(2x^2 + \frac{k^2}{x}\right)^5$ is a. The coefficient of x^2 in the expansion of $(2kx - 1)^4$ is b. (9709/11/M/J/22 number 3)

   (a) Find a and b in terms of the constant k.

   Let's start by finding the term in x^4,

$$\left(2x^2 + \frac{k^2}{x}\right)^5$$

   To get a term in x^4, $2x^2$ has to be raised to the power 3 and $\frac{k^2}{x}$ to the power 2,

$$\binom{5}{2} \times \left(2x^2\right)^3 \times \left(\frac{k^2}{x}\right)^2$$

Simplify,

$$10 \times 8x^6 \times \frac{k^4}{x^2}$$

$$80k^4x^4$$

In the question we are told that the coefficient of x^4 is a,

$$a = 80k^4$$

Now let's find the term in x^2, in,

$$(2kx - 1)^4$$

To get a term in x^2, $2kx$ has to be raised to the power 2 and -1 to the power 2,

$$\binom{4}{2} \times (2kx)^2 \times (-1)^2$$

Simplify,

$$6 \times 4k^2x^2 \times 1$$

$$24k^2x^2$$

In the question we are told that the coefficient of x^2 is b,

$$b = 24k^2$$

Therefore, the final answer is,

$$a = 80k^4 \quad b = 24k^2$$

(b) Given that $a + b = 216$, find the possible values of k.

$$a = 80k^4 \quad b = 24k^2$$

Substitute a and b in the equation,

$$80k^4 + 24k^2 = 216$$

Put all the terms on one side and simplify,

$$80k^4 + 24k^2 - 216 = 0$$

$$10k^4 + 3k^2 - 27 = 0$$

Solve the hidden quadratic,

$$\left(2k^2 - 3\right)\left(5k^2 + 9\right) = 0$$

$$2k^2 - 3 = 0 \quad 5k^2 + 9 = 0$$

Solve for k in both equations,

$$2k^2 = 3 \quad 5k^2 = -9$$

$$k^2 = \frac{3}{2} \quad k^2 = -\frac{9}{5}$$

$$k = \pm\sqrt{\frac{3}{2}} \quad k = \pm\sqrt{-\frac{9}{5}}$$

$$k = \pm\sqrt{\frac{3}{2}} \quad k = \text{No Solution}$$

Therefore, the final answer is,

$$k = \pm\sqrt{\frac{3}{2}}$$

7. The second and third terms of a geometric progression are 10 and 8 respectively. Find the sum to infinity. (9709/12/M/J/22 number 2)

$$\text{2nd term} = 10 \quad \text{3nd term} = 8$$

We can write this using the n-th term formula as,

$$ar = 10 \quad ar^2 = 8$$

Let's solve these two equations simultaneously. Make a the subject of the formula in the first equation,

$$a = \frac{10}{r}$$

Substitute into the second equation,

$$ar^2 = 8$$

$$\left(\frac{10}{r}\right) r^2 = 8$$

$$10r = 8$$

$$r = \frac{4}{5}$$

Evaluate a,

$$a = \frac{10}{r}$$

$$a = \frac{10}{\frac{4}{5}}$$

$$a = 12.5$$

Now let's find the sum to infinity,

$$a = 12.5 \quad r = \frac{4}{5}$$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = \frac{12.5}{1-\frac{4}{5}}$$

$$S_\infty = 62.5$$

Therefore, the final answer is,

$$S_\infty = 62.5$$

8. The first, second and third terms of an arithmetic progression are k, $6k$, $k+6$ respectively. (9709/12/M/J/22 number 4)

   (a) Find the value of the constant k.

   $$\text{first term} = k \quad \text{second term} = 6k \quad \text{third term} = k+6$$

   Using the formula for n-th term, we can rewrite this as,

   $$a = k \quad a+d = 6k \quad a+2d = k+6$$

   We know that a is k. Substitute a with k in the other two equations,

   $$a+d = 6k \quad a+2d = k+6$$

   $$k+d = 6k \quad k+2d = k+6$$

   $$d = 5k \quad 2d = 6$$

   $$d = 5k \quad 2d = 6$$

   $$d = 5k \quad d = 3$$

   Substitute $d = 3$ into the first equation to evaluate k,

   $$d = 5k$$

   $$3 = 5k$$

   $$k = \frac{3}{5}$$

   Therefore, the final answer is,

   $$k = \frac{3}{5}$$

   (b) Find the sum of the first 30 terms of the progression.

   $$a = k \quad d = 3$$

   Evaluate a,

   $$a = \frac{3}{5}$$

The formula for sum of first n terms is,

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

Substitute into the formula,

$$S_{30} = \frac{1}{2}(30)\left(2\left(\frac{3}{5}\right) + (30-1)3\right)$$

$$S_{30} = 1323$$

Therefore, the final answer is,

$$S_{30} = 1323$$

9. The coefficient of x^3 in the expansion of $\left(p + \frac{1}{p}x\right)^4$ is 144. Find the possible values of the constant p. (9709/13/M/J/22 number 1)

$$\left(p + \frac{1}{p}x\right)^4$$

The term in x^3 can be found by raising p to the power 1 and $\frac{1}{p}x$ to the power 3,

$$\binom{4}{3} \times (p) \times \left(\frac{1}{p}x\right)^3$$

Simplify,

$$4 \times p \times \frac{1}{p^3}x^3$$

$$\frac{4}{p^2}x^3$$

In the question we are told that the coefficient of x^3 is 144,

$$\frac{4}{p^2} = 144$$

Multiply through by p^2 to get rid of the denominator,

$$144p^2 = 4$$

$$p^2 = \frac{1}{36}$$

$$p = \pm\sqrt{\frac{1}{36}}$$

$$p = \pm\frac{1}{6}$$

10. An arithmetic progression has first term 4 and common difference d. The sum of the first n terms of the progression 5863. (9709/13/M/J/22 number 3)

    (a) Show that $(n-1)d = \frac{11726}{n} - 8$.

$$a = 4 \quad d = d \quad S_n = 5863$$

The formula for sum of first n terms is,

$$S_n = \frac{1}{2}n(2a + (n-1)d)$$

Substitute,

$$5863 = \frac{1}{2}n(2(4) + (n-1)d)$$

$$5863 = \frac{1}{2}n(8 + (n-1)d)$$

Let's make $(n-1)d$ the subject of the formula. Multiply both sides by 2,

$$11726 = n(8 + (n-1)d)$$

Divide both sides by n,

$$\frac{11726}{n} = 8 + (n-1)d$$

Subtract 8 from both sides,

$$(n-1)d = \frac{11726}{n} - 8$$

Therefore, we have proved that,

$$(n-1)d = \frac{11726}{n} - 8$$

(b) Given that the n-th term is 139, find the values of n and d, giving the value of d as a fraction.

$$a = 4 \quad (n-1)d = \frac{11726}{n} - 8 \quad u_n = 139$$

The formula for n-th term is,

$$u_n = a + (n-1)d$$

$$u_n = 4 + (n-1)d$$

$$139 = 4 + (n-1)d$$

We have two equations that we can solve simultaneously,

$$139 = 4 + (n-1)d \quad (n-1)d = \frac{11726}{n} - 8$$

Substitute $(n-1)d$ in the first equation with $\frac{11726}{n} - 8$,

$$139 = 4 + (n-1)d$$

$$139 = 4 + \frac{11726}{n} - 8$$

Multiply through by n to get rid of the denominator,

$$139n = 4n + 11726 - 8n$$

Simplify,

$$139n = 11726 - 4n$$

Make n the subject of the formula,

$$143n = 11726$$

$$n = 82$$

Evaluate d,

$$139 = 4 + (n-1)d$$
$$139 = 4 + (82-1)d$$
$$139 = 4 + 81d$$
$$81d = 135$$
$$d = \frac{135}{81}$$
$$d = \frac{5}{3}$$

Therefore, the final answer is,

$$n = 82 \quad d = \frac{5}{3}$$

11. A tool for putting fence posts into the ground is called a 'post-rammer'. The distances in millimetres that the post sinks into the ground on each impact of the post-rammer follow a geometric progression. The first three impacts cause the post to sink into the ground by 50mm, 40mm and 32mm respectively. (9709/11/O/N/22 number 7)

    (a) Verify that the 9-th impact is the first time in which the post sinks less than 10mm into the ground.

    $$\text{first term } = 50 \quad \text{second term} = 40 \quad \text{third term} = 32$$

    Using the formula of the n-th term, we can rewrite this as,

    $$a = 50 \quad ar = 40 \quad ar^2 = 32$$

    Let's evaluate r,

    $$r = \frac{40}{50}$$
    $$r = 0.8$$

To check whether the 9-th impact is the first time the post sinks less than 10mm, we need to check both the 8-th and 9-th impact,

$$u_n = ar^{n-1}$$

$$u_8 = 50(0.8)^{8-1}$$

$$u_8 = 10.5$$

$$u_9 = 50(0.8)^{9-1}$$

$$u_9 = 8.39$$

(b) Find, to the nearest millimetre, the total depth of the post in the ground after 20 impacts.

$$a = 50 \quad r = 0.8$$

The question requires us to find the sum of the first 20 impacts,

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

Substitute into the formula and simplify,

$$S_{20} = \frac{50\left(1-(0.8)^{20}\right)}{1-0.8}$$

$$S_{20} = 247.1177$$

Give your answer correct to the nearest millimetre,

$$S_{20} = 247$$

Therefore, the final answer is,

$$S_{20} = 247$$

(c) Find the greatest total depth in the ground which could be theoretically achieved.

$$a = 50 \quad r = 0.8$$

The greatest total depth would be the sum of infinity impacts i.e the sum to infinity,

$$S_\infty = \frac{a}{1-r}$$

Substitute into the formula and simplify,

$$S_\infty = \frac{50}{1-0.8}$$

$$S_\infty = 250$$

Therefore, the final answer is,

$$S_\infty = 250$$

12. A geometric progression is such that the third term is 1764 and the sum of the second and third terms is 3444. Find the 50-th term. (9709/12/O/N/22 number 4)

$$\text{third term} = 1764 \quad S_{2 \text{ and } 3} = 3444$$

Using the formula for n-th term, we can write the second and third terms as,

$$\text{second term} = ar \quad \text{third term} = ar^2$$

The question tells us that,

$$\text{third term} = 1764 \quad S_{2 \text{ and } 3} = 3444$$

$$ar^2 = 1764 \quad ar + ar^2 = 3444$$

We can solve these two equations simultaneously to evaluate a and r. Make a the subject of the formula in the first equation,

$$a = \frac{1764}{r^2}$$

Substitute $a = \frac{1764}{r^2}$ into the second equation,

$$ar + ar^2 = 3444$$

$$\left(\frac{1764}{r^2}\right) r + \left(\frac{1764}{r^2}\right) r^2 = 3444$$

Expand the brackets and simplify,

$$\frac{1764}{r} + 1764 = 3444$$

Multiply through by r to get rid of the denominator,

$$1764 + 1764r = 3444r$$

Make r the subject of the formula,

$$1680r = 1764$$

$$r = \frac{1764}{1680}$$

$$r = 1.05$$

Evaluate a,

$$a = \frac{1764}{(1.05)^2}$$

$$a = 1600$$

Now let's find the 50-th term,

$$u_n = ar^{n-1}$$
$$u_{50} = 1600(1.05)^{50-1}$$
$$u_{50} = 17474.13301$$

Give the answer correct to 3 significant figures,

$$u_{50} = 17500$$

Therefore, the final answer is,

$$u_{50} = 17500$$

13. (a) Find the first three terms in ascending powers of x of the expansion of $(1+2x)^5$. (9709/13/O/N/22 number 3)

$$a = 1 \quad b = 2x \quad n = 5$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^5 + \binom{5}{1}(1)^{5-1}(2x) + \binom{5}{2}(1)^{5-2}(2x)^2$$

Simplify each term,

$$1 + 10x + 40x^2$$

Therefore, the final answer is,

$$1 + 10x + 40x^2$$

(b) Find the first three terms in ascending powers of x of the expansion of $(1 - 3x)^4$.

$$a = 1 \quad b = -3x \quad n = 4$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^4 + \binom{4}{1}(1)^{4-1}(-3x) + \binom{4}{2}(1)^{4-2}(-3x)^2$$

Simplify each term,

$$1 - 12x + 54x^2$$

Therefore, the final answer is,

$$1 - 12x + 54x^2$$

(c) Hence find the coefficient of x^2 in the expansion of $(1+2x)^5(1-3x)^4$.

$$(1+2x)^5(1-3x)^4$$

Substitute each bracket with their respective expansions from (a) and (b),

$$\left(1 + 10x + 40x^2\right)\left(1 - 12x + 54x^2\right)$$

Identify the terms that will multiply to give a term in x^2,

$$1\left(54x^2\right) + 10x(-12x) + 1\left(40x^2\right)$$

Simplify each term,

$$54x^2 - 120x^2 + 40x^2$$

Sum up all the terms,

$$-26x^2$$

The coefficient of x^2 is,

$$-26$$

Therefore, the final answer is,

$$-26$$

14. (a) Find the first three terms in the expansion, in ascending powers of x, of $(1+x)^5$. (9709/12/F/M/21 number 1)

$$a = 1 \quad b = x \quad n = 5$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^5 + \binom{5}{1}(1)^{5-1}(x) + \binom{5}{2}(1)^{5-2}(x)^2$$

Simplify each term,

$$1 + 5x + 10x^2$$

Therefore, the final answer is,

$$1 + 5x + 10x^2$$

(b) Find the first three terms in the expansion, in ascending powers of x, of $(1-2x)^6$.

$$a = 1 \quad b = -2x \quad n = 6$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^6 + \binom{6}{1}(1)^{6-1}(-2x) + \binom{6}{2}(1)^{4-2}(-2x)^2$$

Simplify each term,

$$1 - 12x + 60x^2$$

Therefore, the final answer is,

$$1 - 12x + 60x^2$$

(c) Hence find the coefficient of x^2 in the expansion of $(1 + x)^5(1 - 2x)^6$.

$$(1 + x)^5(1 - 2x)^6$$

Substitute each bracket with their respective expansions from (a) and (b),

$$\left(1 + 5x + 10x^2\right)\left(1 - 12x + 60x^2\right)$$

Identify the terms that will multiply to give a term in x^2,

$$1\left(60x^2\right) + 5x(-12x) + 1\left(10x^2\right)$$

Simplify each term,

$$60x^2 - 60x^2 + 10x^2$$

Sum up all the terms,

$$10x^2$$

The coefficient of x^2 is,

$$10$$

Therefore, the final answer is,

$$10$$

15. The first term of a geometric progression is $\cos\theta$, where $0 < \theta < \frac{1}{2}\pi$. (9709/12/F/M/21 number 9)

(a) For the case where the progression is geometric, the sum to infinity is $\frac{1}{\cos\theta}$.

i. Show that the second term is $\cos\theta\sin^2\theta$.

$$a = \cos\theta \quad S_\infty = \frac{1}{\cos\theta}$$

Let's use the sum to infinity to find r,

$$S_\infty = \frac{a}{1-r}$$

Substitute into the formula,

$$\frac{1}{\cos\theta} = \frac{\cos\theta}{1-r}$$

Now we need to make r the subject of the formula. Let's cross multiply,

$$(1-r)\times 1 = \cos\theta \times \cos\theta$$

Simplify,

$$1-r = \cos^2\theta$$
$$r = 1-\cos^2\theta$$

Let's use the identity $\sin^2\theta \equiv 1-\cos^2\theta$ to simplify,

$$r = \sin^2\theta$$

Now let's find the second term,

$$u_2 = ar$$
$$u_2 = \cos\theta\sin^2\theta$$

Therefore, the final answer is,

$$u_2 = \cos\theta\sin^2\theta$$

ii. Find the sum of the first 12 terms where $\theta = \frac{1}{3}\pi$, giving your answer correct to 4 significant figures.

$$a = \cos\theta \quad r = \sin^2\theta$$

Substitute θ in a and r with $\frac{1}{3}\pi$,

$$a = \cos\frac{1}{3}\pi \quad r = \sin^2\frac{1}{3}\pi$$
$$a = 0.5 \quad r = 0.75$$

Now let's find the sum of the first 12 terms,

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

Substitute into the formula and simplify,

$$S_{12} = \frac{0.5\left(1-(0.75)^{12}\right)}{1-0.75}$$
$$S_{12} = 1.936647296$$

Give your answer correct to 4 significant figures,

$$S_{12} = 1.937$$

Therefore, the final answer is,

$$S_{12} = 1.937$$

(b) For the case where the progression is arithmetic, the first two terms are again $\cos\theta$ and $\cos\theta\sin^2\theta$ respectively. Find the 85-th term when $\theta = \frac{1}{3}\pi$.

$$\text{first term} = \cos\theta \quad \text{second term} = \cos\theta\sin^2\theta \quad \theta = \frac{1}{3}\pi$$

Substitute θ in the first and second terms with $\frac{1}{3}\pi$,

$$\text{first term} = \cos\frac{1}{3}\pi \quad \text{second term} = \cos\left(\frac{1}{3}\pi\right)\sin^2\frac{1}{3}\pi$$

$$\text{first term} = 0.5 \quad \text{second term} = 0.5 \times 0.75$$

$$\text{first term} = 0.5 \quad \text{second term} = 0.375$$

Let's find the common difference,

$$d = 0.375 - 0.5$$

$$d = -0.125$$

Now let's find the 85-th term,

$$u_n = a + (n-1)d$$

Substitute into the formula and simplify,

$$u_{85} = 0.5 + (85-1)(-0.125)$$

$$u_{85} = -10$$

Therefore, the final answer is,

$$u_{85} = -10$$

16. (a) Find the first three terms in the expansion of $(3-2x)^5$ in ascending powers of x. (9709/11/M/J/21 number 3)

$$a = 3 \quad b = -2x \quad n = 5$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$3^5 + \binom{5}{1}(3)^{5-1}(-2x) + \binom{5}{2}(3)^{5-2}(-2x)^2$$

Simplify each term,

$$243 - 810x + 1080x^2$$

Therefore, the final answer is,

$$243 - 810x + 1080x^2$$

(b) Hence find the coefficient of x^2 in the expansion of $(4+x)^2(3-2x)^5$.

$$(3-2x)^5 = 243 - 810x + 1080x^2$$

Let's start by fully expanding $(4+x)^2$,

$$(4+x)^2 = 16 + 8x + x^2$$

Now let's go back to the problem,

$$(4+x)^2(3-2x)^5$$

Substitute each bracket with its respective expansion,

$$\left(16 + 8x + x^2\right)\left(243 - 810x + 1080x^2\right)$$

Identity which terms result in a term in x^2 when multiplied together,

$$16\left(1080x^2\right) + 8x(-810x) + x^2(243)$$

Simplify each term,

$$17280x^2 - 6480x^2 + 243x^2$$

Sum up all the terms,

$$11043x^2$$

Therefore, the final answer is,

$$11043$$

17. The fifth, sixth and seventh terms of a geometric progression are $8k$, -12 and $2k$ respectively. Given that k is negative, find the sum to infinity of the progression. (9709/11/M/J/21 number 5)

$$\text{fifth term} = 8k \quad \text{sixth term} = -12 \quad \text{seventh term} = 2k$$

Using the formula for n-th term we can write the above as,

$$ar^4 = 8k \quad ar^5 = -12 \quad ar^6 = 2k$$

We can find r in two ways,

$$r = \frac{ar^5}{ar^4} \quad r = \frac{ar^6}{ar^5}$$

$$r = \frac{-12}{8k} \quad r = \frac{2k}{-12}$$

$$r = -\frac{3}{2k} \quad r = -\frac{k}{6}$$

Let's equate the two equations to evaluate k,

$$-\frac{3}{2k} = -\frac{k}{6}$$

Multiply through by -1,

$$\frac{3}{2k} = \frac{k}{6}$$

Cross multiply,

$$3(6) = k(2k)$$
$$2k^2 = 18$$

Make k the subject of the formula,

$$k^2 = 9$$
$$k = \pm\sqrt{9}$$
$$k = \pm 3$$

In the question we are told that k is negative, therefore, disregard $k = 3$,

$$k = -3$$

Evaluate r,

$$r = -\frac{k}{6}$$
$$r = -\frac{(-3)}{6}$$
$$r = 0.5$$

Evaluate a,

$$ar^5 = -12$$
$$a(0.5)^5 = -12$$
$$\frac{1}{32}a = -12$$
$$a = -12 \times 32$$
$$a = -384$$

Now that we have the values of a and r,

$$a = -384 \quad r = 0.5$$

Let's find the sum to infinity of the progression,

$$S_\infty = \frac{a}{1-r}$$

Substitute into the formula and simplify,

$$S_\infty = \frac{-384}{1-0.5}$$

$$S_\infty = -768$$

18. (a) Expand $\left(1-\frac{1}{2x}\right)^2$. (9709/11/O/N/21 number 1)

To expand start by squaring the first term,

$$1^2$$

$$1$$

Multiply the power by the first term and the second term,

$$2 \times 1 \times -\frac{1}{2x}$$

$$-\frac{1}{x}$$

Square the second term,

$$\left(-\frac{1}{2x}\right)^2$$

$$\frac{1}{4x^2}$$

Add all three terms together,

$$1 - \frac{1}{x} + \frac{1}{4x^2}$$

Therefore, the final answer is,

$$1 - \frac{1}{x}\frac{1}{4x^2}$$

(b) Find the first four terms in the expansion, in ascending powers of x, of $(1+2x)^6$.

$$a = 1 \quad b = 2x \quad n = 6$$

Use the formula for binomial expansion to expand,

$$a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + ... + b^n$$

Substitute into the formula,

$$1^6 + \binom{6}{1}(1)^{6-1}(2x) + \binom{6}{2}(1)^{6-2}(2x)^2 + \binom{6}{3}(1)^{6-3}(2x)^3$$

Simplify each term,

$$1 + 12x + 60x^2 + 160x^3$$

Therefore, the final answer is,

$$1 + 12x + 60x^2 + 160x^3$$

(c) Hence find the coefficient of x in the expansion of $\left(1 - \frac{1}{2x}\right)^2 (1 + 2x)^6$.

$$\left(1 - \frac{1}{2x}\right)^2 (1 + 2x)^6$$

Substitute each bracket with their respective expansions from (a) and (b),

$$\left(1 - \frac{1}{x} + \frac{1}{4x^2}\right)(1 + 12x + 60x^2 + 160x^3)$$

Identify the terms that will multiply to give a term in x,

$$1\,(12x) + -\frac{1}{x}(60x^2) + \frac{1}{4x^2}\,(160x^3)$$

Simplify each term,

$$12x - 60x + 40x$$

Sum up all the terms,

$$-8x$$

The coefficient of x is,

$$-8$$

Therefore, the final answer is,

$$-8$$