Pure Maths 1

1.7 Differentiation - Easy

GCEALevelMaths9709

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**AS Level**
Component:	**Pure Mathematics 1**
Topic:	**1.7 Differentiation**
Difficulty:	**Easy**

Questions

1. At the point $(4, -1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx} = x^{-\frac{1}{2}} + k$, where k is a constant. (9709/12/F/M/23 number 10acd)

   (a) Show that $k = -2$.
   It is given that the equation of the curve is $y = 2\sqrt{x} - 2x + 3$.

   (b) Find the coordinates of the stationary point.

   (c) Determine the nature of the stationary point.

2. The equation of a curve is $y = 54x - (2x - 7)^3$. (9709/12/M/J/20 number 10)

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   (b) Find the coordinates of each of the stationary points on the curve.

   (c) Determine the nature of each of the stationary points.

3. The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$. (9709/11/M/J/23 number 11abd)

   (a) Find the value of a.

   (b) Determine the nature of this stationary point.
   It is given that the equation of the curve is $y = 2x^3 - 15x^2 + 24x + 1$.

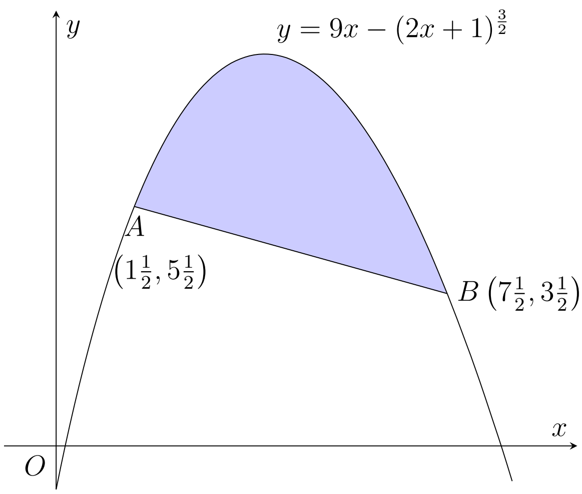
   (c) Find the coordinates of any other stationary point on this curve.

4. The equation of a circle is $(x - a)^2 + (y - 3)^2 = 20$. The line $y = \frac{1}{2}x + 6$ is a tangent to the circle at the point $P(2, 7)$. (9709/12/M/J/23 number 10bc)

   (a) For $a = 4$, find the equation of the normal to the circle at P.

   (b) For $a = 4$, find the equations of the two tangents to the circle which are parallel to the normal found in (b).

5.



The diagram shows the points $A\left(1\frac{1}{2}, 5\frac{1}{2}\right)$ and $B\left(7\frac{1}{2}, 3\frac{1}{2}\right)$ lying on the curve with equation $y = 9x - (2x + 1)^{\frac{3}{2}}$. (9709/13/M/J/23 number 10ab)

   (a) Find the coordinates of the maximum point on the curve.

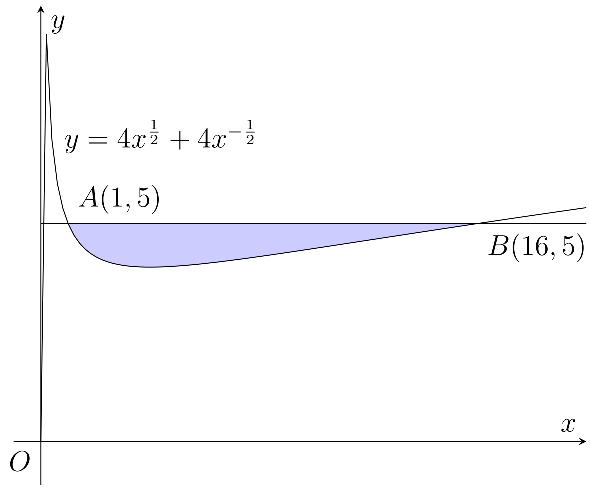
(b) Verify that the line AB is a normal to the curve at A.

6. The equation of a curve is $y = 3x + 1 - 4(3x+1)^{\frac{1}{2}}$ for $x > -\frac{1}{3}$. (9709/12/M/J/22 number 9)

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

   (b) Find the coordinates of the stationary point and determine its nature.

7.



The diagram shows the curve with equation $y = x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$. The line $y = 5$ intersects the curve at the points $A(1,5)$ and $B(16,5)$. (9709/13/M/J/22 number 8a) Find the equation of the tangent to the curve at the point A.

8. The point $P(-1, m)$ lies on the line with equation $y = mx+c$, where m and c are positive constants. A curve has equation $y = -\frac{m}{x}$. There is a single point P on the curve such that the straight line is a tangent to the curve at P. The normal to the curve at P intersects the curve again at the point Q. (9709/13/M/J/22 number 11b) Find the coordinates of Q in terms of m.

9. The equation of a curve is such that $\frac{dy}{dx} = 12\left(\frac{1}{2}x - 1\right)^{-4}$. It is given that the curve passes through the point $P(6,4)$. (9709/11/O/N/22 number 2a) Find the equation of the tangent to the curve at P.

10. The function f is defined by $f(x) = 2 - \frac{3}{4x-p}$ for $x > \frac{p}{4}$, where p is a constant. (9709/11/O/N/22 number 8a) Find $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither.

11. The equation of a curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$. The curve passes through the point $(3,5)$. It is given that the equation of the curve is $y = 2x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 5$. (9709/12/O/N/22 number 8bc)

    (a) Find the x-coordinate of the stationary point.

    (b) State the set of values of x for which y increases as x increases.

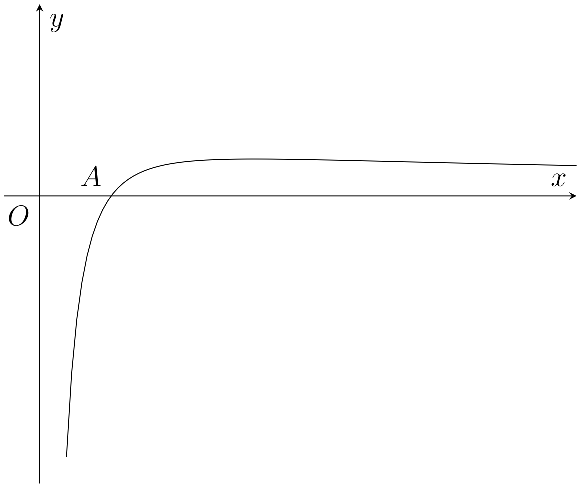
12. (a) Find the coordinates of the minimum point of the curve $y = \frac{9}{4}x^2 - 12x + 18$. (9709/12/O/N/22 number 11ac)

    (b) A point P is moving along the curve $y = 18 - \frac{3}{8}x^{\frac{5}{2}}$ in such a way that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the rate at which the y-coordinate at P is changing when $x = 4$.

13. The curve $y = f(x)$ is such that $f'(x) = \frac{-3}{(x+2)^4}$. (9709/13/O/N/22 number 7a) The tangent at a point on the curve where $x = a$ has a gradient $-\frac{16}{27}$. Find the possible values of a.

14. A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^2}$ and $A(1, -3)$ lies on the curve. A point is moving along the curve and at A the y-coordinate of the point is increasing at 3 units per second. (9709/12/F/M/21 number 6a) Find the rate of increase at A of the x-coordinate of the point.

15.



The diagram shows the curve with equation $y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$. The curve crosses the x-axis at the point A. (9709/12/F/M/21 number 11a-c)

(a) Find the x-coordinate of A.

(b) Find the equation of the tangent to the curve at A.

(c) Find the x-coordinate of the maximum point of the curve.

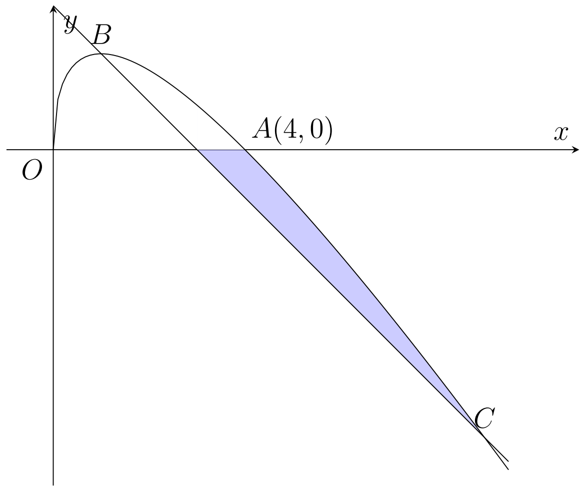
16. The gradient of the curve is given by $\frac{dy}{dx} = 6(3x-5)^3 - kx^2$, where k is a constant. The curve has a stationary point at $(2, -3.5)$. (9709/12/M/J/21 number 11acd)

(a) Find the value of k.
It is given that the equation of the curve is $y = \frac{1}{2}(3x-5)^3 - \frac{1}{2}x^3$.

(b) Find $\frac{d^2y}{dx^2}$.

(c) Determine the nature of the stationary point at $(2, -3.5)$.

17.



The diagram shows a curve with equation $y = 4x^{\frac{1}{2}} - 2x$ for $x \geq 0$, and a straight line with equation $y = 3 - x$. The curve crosses the x-axis at $A(4,0)$ and crosses the straight line at $B(1,2)$ and C. (9709/11/O/N/20 number 12b) Show that B is a stationary point on the curve.

Answers

1. At the point $(4, -1)$ on a curve, the gradient of the curve is $-\frac{3}{2}$. It is given that $\frac{dy}{dx} = x^{-\frac{1}{2}} + k$, where k is a constant. (9709/12/F/M/23 number 10acd)

   (a) Show that $k = -2$.

   Let's find an expression for $\frac{dy}{dx}$ at $(4, -1)$,

$$\frac{dy}{dx} = x^{-\frac{1}{2}} + k$$

$$\frac{dy}{dx} = 4^{-\frac{1}{2}} + k$$

$$\frac{dy}{dx} = \frac{1}{2} + k$$

   At $(4, -1)$ the gradient of the curve is $-\frac{3}{2}$. Let's equate our expression for $\frac{dy}{dx}$ at $(4, -1)$ to $-\frac{3}{2}$,

$$\frac{1}{2} + k = -\frac{3}{2}$$

   Solve for k,

$$k = -\frac{3}{2} - \frac{1}{2}$$

$$k = -2$$

   Therefore, we have proved that,

$$k = -2$$

   It is given that the equation of the curve is $y = 2\sqrt{x} - 2x + 3$.

   (b) Find the coordinates of the stationary point.

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 2$$

   At a stationary point $\frac{dy}{dx}$ is 0,

$$x^{-\frac{1}{2}} - 2 = 0$$

   Solve for x,

$$\frac{1}{x^{\frac{1}{2}}} - 2 = 0$$

$$1 - 2x^{\frac{1}{2}} = 0$$

$$2x^{\frac{1}{2}} = 1$$

$$x^{\frac{1}{2}} = \frac{1}{2}$$

$$x = \left(\frac{1}{2}\right)^2$$

$$x = \frac{1}{4}$$

Evaluate y,

$$y = 2\sqrt{x} - 2x + 3$$
$$y = 2\sqrt{\frac{1}{4}} - 2\left(\frac{1}{4}\right) + 3$$
$$y = \frac{7}{2}$$

The coordinates of the stationary point are,

$$\left(\frac{1}{4}, \frac{7}{2}\right)$$

Therefore, the final answer is,

$$\left(\frac{1}{4}, \frac{7}{2}\right)$$

(c) Determine the nature of the stationary point.

$$\frac{dy}{dx} = x^{-\frac{1}{2}} - 2$$

Let's find the second derivative,

$$\frac{d^2y}{dx^2} = -\frac{1}{2}x^{-\frac{3}{2}}$$

Substitute in the x-value, $\frac{1}{4}$, of the stationary point $\left(\frac{1}{4}, \frac{7}{2}\right)$,

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}}$$
$$\frac{d^2y}{dx^2} = -4$$

Therefore, the final answer is,

The second derivative is negative, therefore, the point $\left(\frac{1}{4}, \frac{7}{2}\right)$ is a maximum point.

2. The equation of a curve is $y = 54x - (2x-7)^3$. (9709/12/M/J/20 number 10)

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$y = 54x - (2x-7)^3$$

Let's start by finding the first derivative,

$$\frac{dy}{dx} = 54 - 3(2x-7)^2 \times 2$$
$$\frac{dy}{dx} = 54 - 6(2x-7)^2$$

Now let's find the second derivative,

$$\frac{d^2y}{dx^2} = -6(2)(2x-7) \times 2$$
$$\frac{d^2y}{dx^2} = -24(2x-7)$$

Therefore, the final answer is,

$$\frac{dy}{dx} = 54 - 6(2x-7)^2 \quad \frac{d^2y}{dx^2} = -24(2x-7)$$

(b) Find the coordinates of each of the stationary points on the curve.

$$\frac{dy}{dx} = 54 - 6(2x-7)^2$$

At a stationary point, $\frac{dy}{dx}$ is 0,

$$54 - 6(2x-7)^2 = 0$$

Solve the quadratic equation,

$$6(2x-7)^2 = 54$$
$$(2x-7)^2 = 9$$
$$2x-7 = \pm\sqrt{9}$$
$$2x-7 = \pm 3$$
$$2x = 7 \pm 3$$
$$x = \frac{7 \pm 3}{2}$$
$$x = 2, 5$$

Evaluate the y-coordinates,

$$y = 54x - (2x-7)^3$$
$$\text{At } x = 2 \quad \text{At } x = 5$$
$$y = 54(2) - (2(2)-7)^3 \quad y = 54(5) - (2(5)-7)^3$$
$$y = 135 \quad y = 243$$

The coordinates of the stationary points are,

$$(2, 135) \quad (5, 243)$$

Therefore, the final answer is,

$$(2, 135) \quad (5, 243)$$

(c) Determine the nature of each of the stationary points.

$$\frac{d^2y}{dx^2} = -24(2x-7)$$

Substitute the x-value at $(2, 135)$ into the second derivative,

$$\frac{d^2y}{dx^2} = -24(2(2) - 7)$$

$$\frac{d^2y}{dx^2} = 72$$

Point $(2, 135)$ is a minimum point since its second derivative is positive. Substitute the x-value at $(5, 243)$ into the second derivative,

$$\frac{d^2y}{dx^2} = -24(2(5) - 7)$$

$$\frac{d^2y}{dx^2} = -72$$

Point $(5, 243)$ is a maximum point since its second derivative is negative.

Therefore, the final answer is,

$(2, 135)$ is a minimum point since its second derivative is positive

$(5, 24)$ is a maximum point since its second derivative is negative

3. The equation of a curve is such that $\frac{dy}{dx} = 6x^2 - 30x + 6a$, where a is a positive constant. The curve has a stationary point at $(a, -15)$. (9709/11/M/J/23 number 11abd)

   (a) Find the value of a.

$$\frac{dy}{dx} = 6x^2 - 30x + 6a$$

At a stationary point, $\frac{dy}{dx}$ is 0,

$$6x^2 - 30x + 6a = 0$$

We are told that the x-value at the stationary point is a,

$$6a^2 - 30a + 6a = 0$$

$$6a^2 - 24a = 0$$

Solve the quadratic equation,

$$6a(a - 4) = 0$$

$$a = 0 \quad a = 4$$

We are told that a is a positive constant,

$$a = 4$$

Therefore, the final answer is,

$$a = 4$$

(b) Determine the nature of this stationary point.

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

Let's start by finding the second derivative,

$$\frac{d^2y}{dx^2} = 2(6)x - 30$$
$$\frac{d^2y}{dx^2} = 12x - 30$$

Let's substitute in the x-value, 4, of the stationary point,

$$\frac{d^2y}{dx^2} = 12(4) - 30$$
$$\frac{d^2y}{dx^2} = 18$$

Since the second derivative is positive, it is a minimum point.

Therefore, the final answer is,

$(4, -15)$ is a minimum point since its second derivative is positive.

It is given that the equation of the curve is $y = 2x^3 - 15x^2 + 24x + 1$.

(c) Find the coordinates of any other stationary point on this curve.

$$\frac{dy}{dx} = 6x^2 - 30x + 24$$

At a stationary point, $\frac{dy}{dx}$ is 0,

$$6x^2 - 30x + 24 = 0$$
$$x^2 - 5x + 4 = 0$$

Solve the quadratic,

$$(x - 4)(x - 1) = 0$$
$$x = 4, 1$$

At the other stationary point,

$$x = 1$$

Let's evaluate y,

$$y = 2x^3 - 15x^2 + 24x + 1$$
$$y = 2(1)^3 - 15(1)^2 + 24(1) + 1$$
$$y = 12$$

Therefore, the coordinates of the other stationary point are,

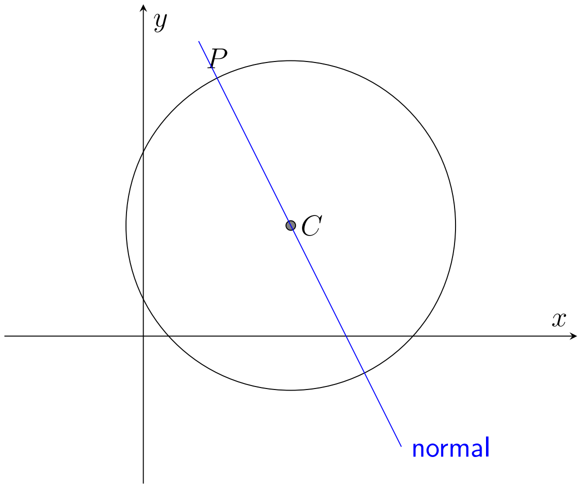
$$(1, 12)$$

4. The equation of a circle is $(x-a)^2 + (y-3)^2 = 20$. The line $y = \frac{1}{2}x + 6$ is a tangent to the circle at the point $P(2, 7)$. (9709/12/M/J/23 number 10bc)

   (a) For $a = 4$, find the equation of the normal to the circle at P.

   $$(x-4)^2 + (y-3)^2 = 20 \quad y = \frac{1}{2}x + 6$$

   Let's sketch a diagram of the problem,

   The normal passes through C and P. So the gradient of the normal is the same as the gradient of CP,

   $$m = \frac{3-7}{4-2}$$
   $$m = -2$$

   The point $P(2, 7)$ lies on the normal. Now let's find the equation of the normal,

   $$y = mx + c \quad m = -2 \quad \text{passing through } P(2, 7)$$
   $$7 = -2(2) + c$$
   $$7 = -4 + c$$
   $$c = 11$$

   The equation of the normal is,

   $$y = -2x + 11$$

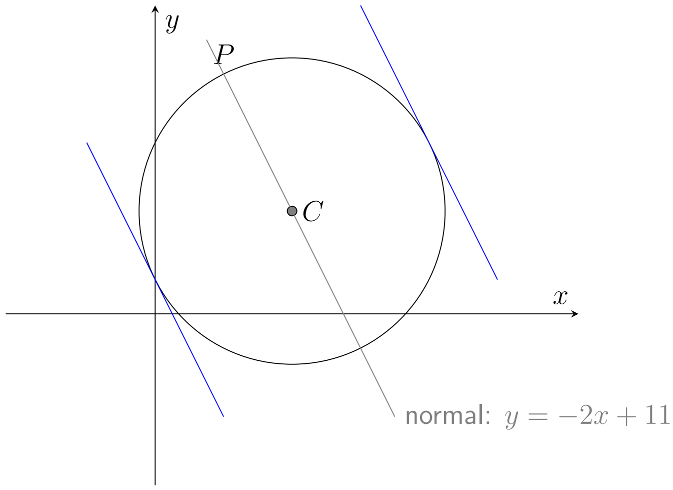
   Therefore, the final answer is,

   $$y = -2x + 11$$

   (b) For $a = 4$, find the equations of the two tangents to the circle which are parallel to the normal found in (b).

   $$(x-4)^2 + (y-3)^2 = 20$$

Let's sketch a diagram of the problem,



The tangents are parallel to the normal, so they have the same gradient as the normal,

$$y = -2x + c$$

Since these lines are tangent to the circle,

$$b^2 - 4ac = 0$$

Let's first solve our two equations simultaneously,

$$(x-4)^2 + (y-3)^2 = 20 \quad y = -2x + c$$

Substitute the linear equation into the equation of the circle,

$$(x-4)^2 + (-2x + c - 3)^2 = 20$$

$$(x-4)^2 + (-2x + (c-3))^2 = 20$$

Expand the brackets and simplify,

$$x^2 - 8x + 16 + 4x^2 + 2(-2x)(c-3) + (c-3)^2 = 20$$
$$x^2 + 4x^2 - 8x + (-4c + 12)x + (c-3)^2 + 16 - 20 = 0$$
$$5x^2 + (-4c + 12 - 8)x + (c-3)^2 - 4 = 0$$
$$5x^2 + (-4c + 4)x + (c-3)^2 - 4 = 0$$

Identity a, b and c,

$$a = 5 \quad b = -4c + 4 \quad c = (c-3)^2 - 4$$

Substitute into the discriminant,

$$b^2 - 4ac = 0$$

$$(-4c+4)^2 - 4(5)\left[(c-3)^2 - 4\right] = 0$$

Expand the brackets and simplify,

$$16c^2 - 32c + 16 - 20\left[c^2 - 6c + 9 - 4\right] = 0$$

$$16c^2 - 32c + 16 - 20\left(c^2 - 6c + 5\right) = 0$$

$$16c^2 - 32c + 16 - 20c^2 + 120c - 100 = 0$$

$$-4c^2 + 88c - 84 = 0$$

$$c^2 - 22c + 21 = 0$$

Solve the quadratic equation,

$$(c - 21)(c - 1) = 0$$

$$c = 21 \quad c = 1$$

The equations of the tangents are,

$$y = -2x + c$$

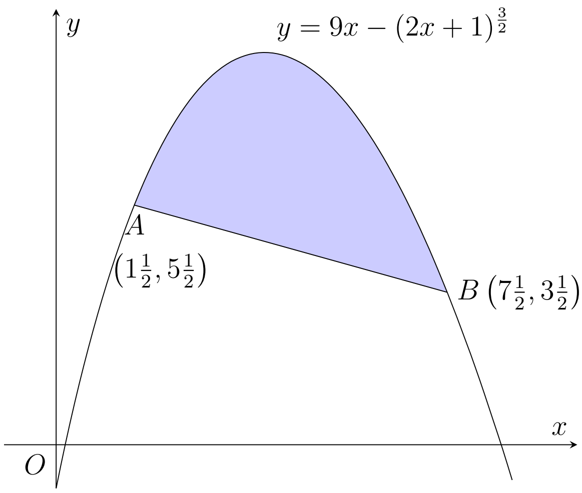
At $c = 21$ At $c = 1$

$$y = -2x + 21 \quad y = -2x + 1$$

Therefore, the final answer is,

$$y = -2x + 21 \quad y = -2x + 1$$

5.



The diagram shows the points $A\left(1\frac{1}{2}, 5\frac{1}{2}\right)$ and $B\left(7\frac{1}{2}, 3\frac{1}{2}\right)$ lying on the curve with equation $y = 9x - (2x+1)^{\frac{3}{2}}$. (9709/13/M/J/23 number 10ab)

(a) Find the coordinates of the maximum point on the curve.

$$y = 9x - (2x+1)^{\frac{3}{2}}$$

At a maximum point, $\frac{dy}{dx} = 0$. Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx} = 9 - \frac{3}{2}(2x+1)^{\frac{1}{2}} \times 2$$

$$\frac{dy}{dx} = 9 - 3(2x+1)^{\frac{1}{2}}$$

Equate $\frac{dy}{dx}$ to 0 and solve for x,

$$9 - 3(2x+1)^{\frac{1}{2}} = 0$$

$$3(2x+1)^{\frac{1}{2}} = 9$$

$$(2x+1)^{\frac{1}{2}} = 3$$

$$2x + 1 = 9$$

$$2x = 8$$

$$x = 4$$

Evaluate y,

$$y = 9x - (2x+1)^{\frac{3}{2}}$$

$$y = 9(4) - (2(4)+1)^{\frac{3}{2}}$$

$$y = 9$$

Therefore, the coordinates of the maximum point are,

$$(4, 9)$$

(b) Verify that the line AB is a normal to the curve at A.

$$\frac{dy}{dx} = 9 - 3(2x+1)^{\frac{1}{2}}$$

Let's first find the gradient of the curve at $A\left(1\frac{1}{2}, 5\frac{1}{2}\right)$,

$$\frac{dy}{dx} = 9 - 3(2(1\frac{1}{2}) + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 3$$

$$m_{\text{tangent}} = 3$$

If AB is a normal to the curve at A,

$$m_{\text{tangent}} \times m_{AB} = -1$$

Let's find the gradient of AB,

$$m_{AB} = \frac{5\frac{1}{2} - 3\frac{1}{2}}{1\frac{1}{2} - 7\frac{1}{2}}$$

$$m_{AB} = -\frac{1}{3}$$

Let's check if AB is the normal to the curve at A,

$$m_{\text{tangent}} \times m_{AB} = -1$$

$$3 \times -\frac{1}{3} = -1$$

Therefore, since,

$$m_{\text{tangent}} \times m_{AB} = -1$$

We have proved that AB is a normal to the curve at A.

6. The equation of a curve is $y = 3x + 1 - 4(3x+1)^{\frac{1}{2}}$ for $x > -\frac{1}{3}$. (9709/12/M/J/22 number 9)

   (a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$y = 3x + 1 - 4(3x+1)^{\frac{1}{2}}$$

Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx} = 3 - \frac{1}{2}(4)(3x+1)^{-\frac{1}{2}} \times 3$$

$$\frac{dy}{dx} = 3 - 6(3x+1)^{-\frac{1}{2}}$$

Now let's find $\frac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2} = -6\left(-\frac{1}{2}\right)(3x+1)^{-\frac{3}{2}} \times 3$$

$$\frac{d^2y}{dx^2} = 9(3x+1)^{-\frac{3}{2}}$$

Therefore, the final answer is,

$$\frac{dy}{dx} = 3 - 6(3x+1)^{-\frac{1}{2}} \quad \frac{d^2y}{dx^2} = 9(3x+1)^{-\frac{3}{2}}$$

   (b) Find the coordinates of the stationary point and determine its nature.

$$y = 3x + 1 - 4(3x+1)^{\frac{1}{2}} \quad \frac{dy}{dx} = 3 - 6(3x+1)^{-\frac{1}{2}} \quad \frac{d^2y}{dx^2} = 9(3x+1)^{-\frac{3}{2}}$$

At a stationary point, $\frac{dy}{dx} = 0$,

$$3 - 6(3x+1)^{-\frac{1}{2}} = 0$$

Solve for x,

$$6(3x+1)^{-\frac{1}{2}} = 3$$
$$\frac{6}{(3x+1)^{\frac{1}{2}}} = 3$$
$$(3x+1)^{\frac{1}{2}} = \frac{6}{3}$$
$$(3x+1)^{\frac{1}{2}} = 2$$
$$3x+1 = 2^2$$
$$3x+1 = 4$$
$$3x = 3$$
$$x = 1$$

Evaluate y,

$$y = 3x+1-4(3x+1)^{\frac{1}{2}}$$
$$y = 3(1)+1-4(3(1)+1)^{\frac{1}{2}}$$
$$y = -4$$

The coordinates of the stationary point are,

$$(1, -4)$$

Let's evaluate its nature,

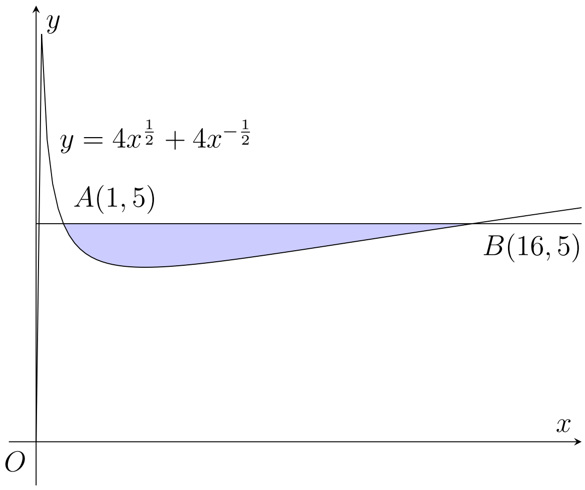
$$\frac{d^2y}{dx^2} = 9(3x+1)^{-\frac{3}{2}}$$
$$\frac{d^2y}{dx^2} = 9(3(1)+1)^{-\frac{3}{2}}$$
$$\frac{d^2y}{dx^2} = 1.125$$

The stationary point is a minimum point since its second derivative is positive.

Therefore, the final answer is,

$$(1, -4) \ \text{ It is a minimum point since its second derivative is positive.}$$

7.



The diagram shows the curve with equation $y = x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$. The line $y = 5$ intersects the curve at the points $A(1, 5)$ and $B(16, 5)$. (9709/13/M/J/22 number 8a) Find the equation of the tangent to the curve at the point A.

$$y = x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}$$

Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}(4)x^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}$$

Let's find the gradient of the tangent at $A(1, 5)$,

$$\frac{dy}{dx} = \frac{1}{2}(1)^{-\frac{1}{2}} - 2(1)^{-\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{3}{2}$$

The gradient of the tangent to the curve at A is,

$$m = -\frac{3}{2}$$

Now let's find the equation of the tangent at A,

$$y = mx + c \quad m = -\frac{3}{2} \quad \text{passing through } A(1, 5)$$

$$5 = -\frac{3}{2}(1) + c$$

$$5 = -\frac{3}{2} + c$$

$$c = 5 + \frac{3}{2}$$

$$c = \frac{13}{2}$$

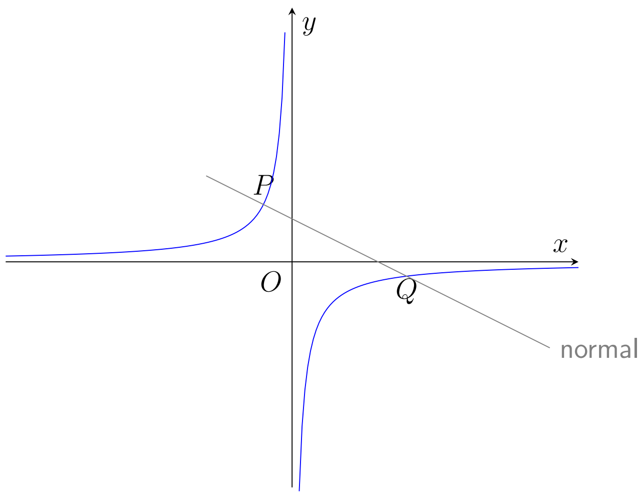
Therefore, the equation of the tangent to the curve at A is,

$$y = -\frac{3}{2}x + \frac{13}{2}$$

8. The point $P(-1, m)$ lies on the line with equation $y = mx + c$, where m and c are positive constants. A curve has equation $y = -\frac{m}{x}$. There is a single point P on the curve such that the straight line is a tangent to the curve at P. The normal to the curve at P intersects the curve again at the point Q. (9709/13/M/J/22 number 11b) Find the coordinates of Q in terms of m.

$$y = -\frac{m}{x} \quad P(-1, m)$$

Let's sketch a diagram of the problem,



Q is the intersection of the normal and the curve. We need to solve the equations of the normal and the curve simultaneously. Let's first find the equation of the normal. The equation of the tangent at P is,

$$y = mx + c$$

Since the gradient of the tangent is m the gradient of the normal must be the negative reciprocal,

$$m_{\text{normal}} = -\frac{1}{m}$$

Now let's find the equation of the normal,

$$y = mx + c \quad m = -\frac{1}{m} \ \text{ Passing through } P(-1, m)$$

$$m = \left(-\frac{1}{m}\right)(-1) + c$$

$$m = \frac{1}{m} + c$$

$$c = m - \frac{1}{m}$$

$$c = \frac{m^2 - 1}{m}$$

The equation of the normal is,

$$y = -\frac{1}{m}x + \frac{m^2 - 1}{m}$$

Now we can solve the equation of the curve and the normal simultaneously,

$$y = -\frac{m}{x} \quad y = -\frac{1}{m}x + \frac{m^2 - 1}{m}$$

$$-\frac{m}{x} = -\frac{1}{m}x + \frac{m^2 - 1}{m}$$

Multiply through by mx to get rid of the denominators,

$$-m^2 = -x^2 + x\left(m^2 - 1\right)$$

Put all the terms on one side,

$$x^2 - \left(m^2 - 1\right)x - m^2 = 0$$

Solve the quadratic by factorisation,

$$(x + 1)\left(x - m^2\right) = 0$$

$$x = -1 \quad x = m^2$$

-1 is the x-coordinate for P, so the x-coordinate for Q is,

$$x = m^2$$

Evaluate the y-coordinate of Q,

$$y = -\frac{m}{x}$$

$$y = -\frac{m}{m^2}$$

$$y = -\frac{1}{m}$$

Therefore, the coordinates of Q are,

$$\left(m^2, -\frac{1}{m}\right)$$

9. The equation of a curve is such that $\frac{dy}{dx} = 12\left(\frac{1}{2}x - 1\right)^{-4}$. It is given that the curve passes through the point $P(6, 4)$. (9709/11/O/N/22 number 2a) Find the equation of the tangent to the curve at P.

$$\frac{dy}{dx} = 12\left(\frac{1}{2}x - 1\right)^{-4}$$

Let's find the gradient of the tangent at P,

$$\frac{dy}{dx} = 12\left(\frac{1}{2}(6) - 1\right)^{-4}$$
$$\frac{dy}{dx} = \frac{3}{4}$$

Now let's find the equation of the tangent to the curve at P,

$$y = mx + c \quad m = \frac{3}{4} \text{ passing through } P(6,4)$$
$$4 = \frac{3}{4}(6) + c$$
$$4 = \frac{9}{2} + c$$
$$c = -\frac{1}{2}$$

Therefore, the equation of the tangent to the curve at P is,

$$y = \frac{3}{4}x - \frac{1}{2}$$

10. The function f is defined by $f(x) = 2 - \frac{3}{4x-p}$ for $x > \frac{p}{4}$, where p is a constant. (9709/11/O/N/22 number 8a) Find $f'(x)$ and hence determine whether f is an increasing function, a decreasing function or neither.

$$f(x) = 2 - 3(4x-p)^{-1}$$

Let's find $f'(x)$,

$$f'(x) = -(-1)(3)(4x-p)^{-2} \times 4$$
$$f'(x) = 12(4x-p)^{-2}$$

Take the bracket to the denominator, using laws of indices, to make the power positive,

$$f'(x) = \frac{12}{(4x-p)^2}$$

Let's inspect the denominator of $f'(x)$. Since it is squared, it will always be positive for all values of x. The numerator of $f'(x)$ is positive. This means that $f'(x)$ is positive for all values of x i.e the gradient is always positive. For that reason it must be an increasing function.

Therefore, the final answer is,

$$f'(x) = \frac{12}{(4x-p)^2}$$

f is an increasing function, since its gradient $f'(x)$ is always positive.

11. The equation of a curve is such that $\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$. The curve passes through the point $(3,5)$. It is given that the equation of the curve is $y = 2x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + 5$. (9709/12/O/N/22 number 8bc)

    (a) Find the x-coordinate of the stationary point.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$$

At a stationary point, $\frac{dy}{dx} = 0$,

$$3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0$$

Make all the powers of x positive,

$$3x^{\frac{1}{2}} - \frac{3}{x^{\frac{1}{2}}} = 0$$

Multiply through by $x^{\frac{1}{2}}$ to get rid of the denominator and simplify,

$$3x^{\frac{1}{2}} \times x^{\frac{1}{2}} - 3 = 0$$

$$3x - 3 = 0$$

Make x the subject of the formula,

$$3x = 3$$

$$x = 1$$

Therefore, the x-coordinate of the stationary point is,

$$x = 1$$

(b) State the set of values of x for which y increases as x increases.

$$\frac{dy}{dx} = 3x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$$

If the gradient is positive, it means that x and y values are increasing. We know that the turning point of the function is at $x = 1$. So we need to check either side of the turning point to see where the values of the gradient are positive. Let's select a value less than 1, e.g 0.5,

$$\frac{dy}{dx} = 3(0.5)^{\frac{1}{2}} - 3(0.5)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -2.12...$$

Let's select a value greater than 1, e.g 2,

$$\frac{dy}{dx} = 3(2)^{\frac{1}{2}} - 3(2)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 2.12...$$

The gradient is positive for values greater than 1. This means that for $x > 1$, y increases as x increases.

Therefore, the final answer is,

$$x > 1$$

12. (a) Find the coordinates of the minimum point of the curve $y = \frac{9}{4}x^2 - 12x + 18$. (9709/12/O/N/22 number 11ac)

$$y = \frac{9}{4}x^2 - 12x + 18$$

Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{9}{4}(2)x - 12$$
$$\frac{dy}{dx} = \frac{9}{2}x - 12$$

At a minimum point, $\frac{dy}{dx} = 0$,

$$\frac{9}{2}x - 12 = 0$$

Solve for x,

$$\frac{9}{2}x = 12$$
$$x = 12 \times \frac{2}{9}$$
$$x = \frac{8}{3}$$

Evaluate y,

$$y = \frac{9}{4}x^2 - 12x + 18$$
$$y = \frac{9}{4}\left(\frac{8}{3}\right)^2 - 12\left(\frac{8}{3}\right) + 18$$
$$y = 2$$

Therefore, the coordinates of the minimum point are,

$$\left(\frac{8}{3}, 2\right)$$

(b) A point P is moving along the curve $y = 18 - \frac{3}{8}x^{\frac{5}{2}}$ in such a way that the x-coordinate of P is increasing at a constant rate of 2 units per second. Find the rate at which the y-coordinate at P is changing when $x = 4$.

$$y = 18 - \frac{3}{8}x^{\frac{5}{2}} \quad \frac{dx}{dt} = 2 \quad \frac{dy}{dt} = ? \quad x = 4$$

Using the equations above we can construct a chain rule to find $\frac{dy}{dt}$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

We already have the value of $\frac{dx}{dt}$,

$$\frac{dy}{dt} = \frac{dy}{dx} \times 2$$

Let's find $\frac{dy}{dx}$,

$$y = 18 - \frac{3}{8}x^{\frac{5}{2}}$$

$$\frac{dy}{dx} = -\frac{3}{8}\left(\frac{5}{2}\right)x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{15}{16}x^{\frac{3}{2}}$$

Evaluate $\frac{dy}{dx}$ at $x = 4$,

$$\frac{dy}{dx} = -\frac{15}{16}(4)^{\frac{3}{2}}$$

$$\frac{dy}{dx} = -\frac{15}{2}$$

Substitute it into the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \times 2$$

$$\frac{dy}{dt} = -\frac{15}{2} \times 2$$

$$\frac{dy}{dt} = -15$$

Therefore, the y-coordinate is increasing at a rate of,

$$-15 \text{ units per second}$$

13. The curve $y = f(x)$ is such that $f'(x) = \frac{-3}{(x+2)^4}$. (9709/13/O/N/22 number 7a) The tangent at a point on the curve where $x = a$ has a gradient $-\frac{16}{27}$. Find the possible values of a.

$$f'(x) = \frac{-3}{(x+2)^4}$$

We are told that at $x = a$, $f'(x) = -\frac{16}{27}$,

$$f'(a) = -\frac{16}{27}$$

$$\frac{-3}{(a+2)^4} = -\frac{16}{27}$$

Cross multiply,

$$-3(27) = -16(a+2)^4$$

$$-81 = -16(a+2)^4$$

Divide both sides by -16,

$$(a+2)^4 = \frac{81}{16}$$

Take the fourth root of both sides,

$$a+2 = \pm\sqrt[4]{\frac{81}{16}}$$

$$a+2 = \pm\frac{3}{2}$$

$$a = -2 \pm \frac{3}{2}$$

$$a = -\frac{7}{2}, -\frac{1}{2}$$

Therefore, the final answer is,

$$a = -\frac{7}{2}, -\frac{1}{2}$$

14. A curve is such that $\frac{dy}{dx} = \frac{6}{(3x-2)^2}$ and $A(1,-3)$ lies on the curve. A point is moving along the curve and at A the y-coordinate of the point is increasing at 3 units per second. (9709/12/F/M/21 number 6a) Find the rate of increase at A of the x-coordinate of the point.

$$\frac{dy}{dx} = \frac{6}{(3x-2)^2} \quad \frac{dy}{dt} = 3 \quad \frac{dx}{dt} = ? \quad A(1,-3)$$

Let's construct a chain rule for $\frac{dx}{dt}$,

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

We already have the value for $\frac{dy}{dt}$,

$$\frac{dx}{dt} = \frac{dx}{dy} \times 3$$

Let's find $\frac{dx}{dy}$ at $A(1,-3)$ by first finding $\frac{dy}{dx}$ at $A(1,-3)$,

$$\frac{dy}{dx} = \frac{6}{(3x-2)^2}$$

$$\frac{dy}{dx} = \frac{6}{(3(1)-2)^2}$$

$$\frac{dy}{dx} = 6$$

To get $\frac{dx}{dy}$ is the reciprocal of $\frac{dy}{dx}$,

$$\frac{dx}{dy} = \frac{1}{6}$$

Let's substitute into our chain rule,

$$\frac{dx}{dt} = \frac{dx}{dy} \times 3$$

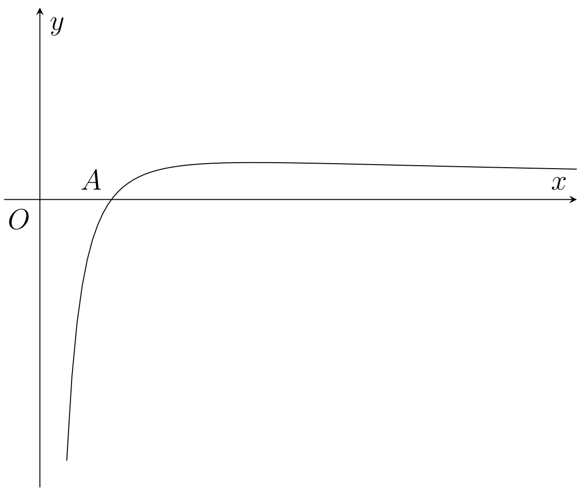
$$\frac{dx}{dt} = \frac{1}{6} \times 3$$

$$\frac{dx}{dt} = \frac{1}{2}$$

Therefore, the rate of increase at A of the x-coordinate of the point is,

$$\frac{1}{2} \text{ units per second}$$

15.



The diagram shows the curve with equation $y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$. The curve crosses the x-axis at the point A. (9709/12/F/M/21 number 11a-c)

(a) Find the x-coordinate of A.

$$y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$$

Since the curve crosses the x-axis at A, we know that the y-coordinate of A is 0. Let's find the x-coordinate of A,

$$y = 9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right)$$

$$9\left(x^{-\frac{1}{2}} - 4x^{-\frac{3}{2}}\right) = 0$$

Make the powers positive,

$$9\left(\frac{1}{x^{\frac{1}{2}}} - \frac{4}{x^{\frac{3}{2}}}\right) = 0$$

Factor out $\frac{1}{x^{\frac{3}{2}}}$ to get rid of the denominators,

$$\frac{9}{x^{\frac{3}{2}}}(x-4)=0$$

$$\frac{9}{x^{\frac{3}{2}}}=0 \quad x-4=0$$

$$x^{\frac{3}{2}}=\frac{9}{0} \quad x=4$$

$$x=\text{No Solutions} \quad x=4$$

$$x=4$$

Therefore, the x-coordinate of A is,

$$x=4$$

(b) Find the equation of the tangent to the curve at A.

$$y=9\left(x^{-\frac{1}{2}}-4x^{-\frac{3}{2}}\right)$$

Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx}=9\left(-\frac{1}{2}x^{-\frac{3}{2}}-4\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}\right)$$

$$\frac{dy}{dx}=9\left(-\frac{1}{2}x^{-\frac{3}{2}}+6x^{-\frac{5}{2}}\right)$$

Find the gradient of the tangent to the curve at $A(4,0)$,

$$\frac{dy}{dx}=9\left(-\frac{1}{2}(4)^{-\frac{3}{2}}+6(4)^{-\frac{5}{2}}\right)$$

$$\frac{dy}{dx}=\frac{9}{8}$$

Now let's find the equation of the tangent to the curve at $A(4,0)$,

$$y=mx+c \quad m=\frac{9}{8} \text{ passing through } A(4,0)$$

$$0=\frac{9}{8}(4)+c$$

$$0=\frac{9}{2}+c$$

$$c=-\frac{9}{2}$$

Therefore, the equation of the tangent to the curve at A is,

$$y = \frac{9}{8}x - \frac{9}{2}$$

(c) Find the x-coordinate of the maximum point of the curve.

$$\frac{dy}{dx} = 9\left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x^{-\frac{5}{2}}\right)$$

At a maximum point, $\frac{dy}{dx} = 0$,

$$9\left(-\frac{1}{2}x^{-\frac{3}{2}} + 6x^{-\frac{5}{2}}\right) = 0$$

Make all the powers positive,

$$9\left(-\frac{1}{2x^{\frac{3}{2}}} + \frac{6}{x^{\frac{5}{2}}}\right) = 0$$

Factor out $\frac{1}{2x^{\frac{5}{2}}}$ and solve for x,

$$\frac{9}{2x^{\frac{5}{2}}}(-x + 12) = 0$$

$$\frac{9}{2x^{\frac{5}{2}}} = 0 \quad -x + 12 = 0$$

$$x = \text{No Solutions} \quad x = 12$$

$$x = 12$$

Therefore, the x-coordinate of the maximum point of the curve is,

$$x = 12$$

16. The gradient of the curve is given by $\frac{dy}{dx} = 6(3x-5)^3 - kx^2$, where k is a constant. The curve has a stationary point at $(2, -3.5)$. (9709/12/M/J/21 number 11acd)

(a) Find the value of k.

$$\frac{dy}{dx} = 6(3x-5)^3 - kx^2$$

We are told that the coordinates of the stationary point are $(2, -3.5)$. We know that at a stationary point $\frac{dy}{dx} = 0$,

$$6(3x-5)^3 - kx^2 = 0$$

The x-coordinate of the stationary point is 2,

$$6(3(2)-5)^3 - k(2)^2 = 0$$

$$6 - 4k = 0$$

Solve for k,

$$4k = 6$$
$$k = \frac{3}{2}$$

Therefore, the final answer is,

$$k = \frac{3}{2}$$

It is given that the equation of the curve is $y = \frac{1}{2}(3x-5)^3 - \frac{1}{2}x^3$.

(b) Find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = 6(3x-5)^3 - \frac{3}{2}x^2$$

Let's find the second derivative,

$$\frac{d^2y}{dx^2} = 3(6)(3x-5)^2 \times 3 - \frac{3}{2}(2)x$$
$$\frac{d^2y}{dx^2} = 54(3x-5)^2 - 3x$$

Therefore, the final answer is,

$$\frac{d^2y}{dx^2} = 54(3x-5)^2 - 3x$$

(c) Determine the nature of the stationary point at $(2, -3.5)$.

$$\frac{d^2y}{dx^2} = 54(3x-5)^2 - 3x$$

Substitute the x-coordinate of the stationary point into the second derivative,

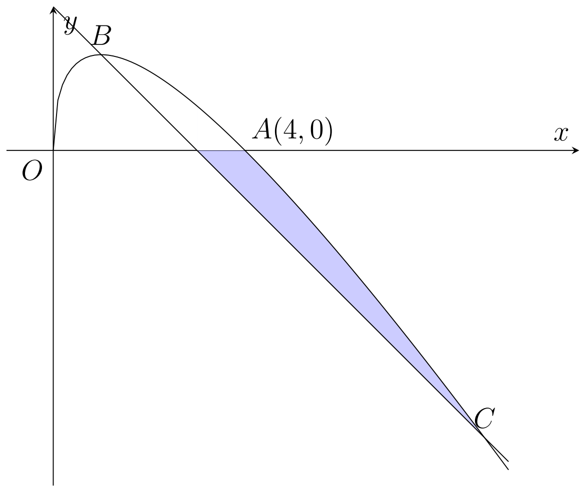
$$\frac{d^2y}{dx^2} = 54(3(2)-5)^2 - 3(2)$$
$$\frac{d^2y}{dx^2} = 48$$

It is a minimum point since the second derivative is positive.

Therefore, the final answer is,

$(2, -3.5)$ is a minimum point because its second derivative is positive.

17.



The diagram shows a curve with equation $y = 4x^{\frac{1}{2}} - 2x$ for $x \geq 0$, and a straight line with equation $y = 3 - x$. The curve crosses the x-axis at $A(4,0)$ and crosses the straight line at $B(1,2)$ and C. (9709/11/O/N/20 number 12b) Show that B is a stationary point on the curve.

$$y = 4x^{\frac{1}{2}} - 2x$$

Let's start by finding $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{1}{2}(4)x^{-\frac{1}{2}} - 2$$
$$\frac{dy}{dx} = 2x^{-\frac{1}{2}} - 2$$
$$\frac{dy}{dx} = \frac{2}{x^{\frac{1}{2}}} - 2$$

At a stationary point, $\frac{dy}{dx} = 0$,

$$\frac{2}{x^{\frac{1}{2}}} - 2 = 0$$

Multiply through by $x^{\frac{1}{2}}$ to get rid of the denominator,

$$2 - 2x^{\frac{1}{2}} = 0$$

Solve for x,

$$2x^{\frac{1}{2}} = 2$$
$$x^{\frac{1}{2}} = 1$$
$$x = \pm 1$$

When $\frac{dy}{dx} = 0$, $x = 1$. In the stem of the question we are told that the x-coordinate of B is 1. Therefore, since we get $x = 1$ as solution from $\frac{dy}{dx} = 0$, B is a stationary point.