Pure Maths 3

3.2 Logarithmic and Exponential Functions - Easy

SMTYL

GCEALevelMaths9709

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**A2 Level**
Component:	**Pure Mathematics 3**
Topic:	**3.2 Logarithmic and Exponential Functions**
Difficulty:	**Easy**

Questions

1. Solve the equation $\ln 3 + \ln(2x + 5) = 2\ln(x + 2)$. Give your answer in a simplified exact form. (9709/32/F/M/20 number 2)
2. Find the set of values of x for which $2\left(3^{1-2x}\right) < 5^x$. Give your answer in a simplified exact form. (9709/31/M/J/20 number 1)
3. (a) Show that the equation
$$\ln\left(1 + e^{-x}\right) + 2x = 0$$
can be expressed as a quadratic equation in e^x. (9709/33/M/J/20 number 3)
   (b) Hence solve the equation $\ln\left(1 + e^{-x}\right) + 2x = 0$, giving your answer correct to 3 decimal places.
4. Solve the equation
$$\ln\left(1 + e^{-3x}\right) = 2$$
Give the answer correct to 3 decimal places. (9709/32/O/N/20 number 1)
5. The variables x and y satisfy the relation $2^y = 3^{1-2x}$. (9709/32/O/N/20 number 3)
   (a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line.
   (b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers.
6. Solve the equation $\ln\left(x^3 - 3\right) = 3\ln x - \ln 3$. Give your answer correct to 3 significant figures. (9709/32/F/M/21 number 1)
7. It is given that $x = \ln(2y - 3) - \ln(y + 4)$.
Express y in terms of x. (9709/32/F/M/23 number 1)
8. Solve the equation
$$3e^{2x} - 4e^{-2x} = 5$$
Give the answer correct to 3 decimal places. (9709/31/M/J/23 number 1)
9. Solve the equation $\ln\left(2x^2 - 3\right) = 2\ln x - \ln 2$, giving your answer in an exact form. (9709/32/M/J/23 number 2)

Answers

1. Solve the equation $\ln 3 + \ln(2x+5) = 2\ln(x+2)$. Give your answer in a simplified exact form. (9709/32/F/M/20 number 2)

$$\ln 3 + \ln(2x+5) = 2\ln(x+2)$$

Let's put all the terms on one side,

$$\ln 3 + \ln(2x+5) - 2\ln(x+2) = 0$$

Rewrite the third term using the laws of logarithms,

$$\ln 3 + \ln(2x+5) - \ln\left((x+2)^2\right) = 0$$

Combine all three terms using the laws of logarithms,

$$\ln\left(\frac{3(2x+5)}{(x+2)^2} = 0\right)$$

Apply the exponential function to both sides to get rid of the $\ln$**,**

$$\frac{3(2x+5)}{(x+2)^2} = e^0$$

$$\frac{3(2x+5)}{(x+2)^2} = 1$$

Cross multiply,

$$3(2x+5) = (x+2)^2$$

Expand the brackets and solve for x**,**

$$6x + 15 = x^2 + 4x + 4$$

$$x^2 + 4x - 6x + 4 - 15 = 0$$

$$x^2 - 2x - 11 = 0$$

$$x = 1 \pm 2\sqrt{3}$$

Remember that the $\ln$ **function cannot take negative values. When** $x = 1 - 2\sqrt{3}$**, it will result in negative values being passed into the** $\ln$ **function, hence we disregard that solution,**

$$x = 1 + 2\sqrt{3}$$

Therefore, the final answer is,

$$x = 1 + 2\sqrt{3}$$

2. Find the set of values of x for which $2\left(3^{1-2x}\right) < 5^x$. Give your answer in a simplified exact form. (9709/31/M/J/20 number 1)

$$2\left(3^{1-2x}\right) < 5^x$$

Apply $\log_5$ on both sides,

$$\log_5\left(2\left(3^{1-2x}\right)\right) < x$$

Use laws of logarithms to split the log on the left-hand side,

$$\log_5 2 + \log_5 3^{1-2x} < x$$

Use laws of logarithms to bring down the power $1 - 2x$,

$$\log_5 2 + (1-2x)\log_5 3 < x$$

Expand the bracket,

$$\log_5 2 + \log_5 3 - 2x\log_5 3 < x$$

Put all the terms containing x on one side,

$$x + 2x\log_5 3 > \log_5 2 + \log_5 3$$

Factor out x on the left-hand side,

$$x\left(1 + 2\log_5 3\right) > \log_5 2 + \log_5 3$$

Use laws of logarithms to simplify the right-hand side,

$$x\left(1 + 2\log_5 3\right) > \log_5(2 \times 3)$$

$$x\left(1 + 2\log_5 3\right) > \log_5(6)$$

Make x the subject of the formula,

$$x > \frac{\log_5 6}{1 + 2\log_5 3}$$

Use laws of indices to simplify $2\log_5 3$,

$$x > \frac{\log_5 6}{1 + \log_5 3^2}$$

$$x > \frac{\log_5 6}{1 + \log_5 9}$$

We can write 1 as $\log_5 5$,

$$x > \frac{\log_5 6}{\log_5 5 + \log_5 9}$$

Use laws of logarithms to simplify the denominator,

$$x > \frac{\log_5 6}{\log_5(5 \times 9)}$$

$$x > \frac{\log_5 6}{\log_5 45}$$

Therefore, the final answer is,

$$x > \frac{\log_5 6}{\log_5 45}$$

3. (a) Show that the equation

$$\ln\left(1+e^{-x}\right)+2x=0$$

can be expressed as a quadratic equation in e^x. (9709/33/M/J/20 number 3)

$$\ln\left(1+e^{-x}\right)+2x=0$$

Move $2x$ to the right-hand side,

$$\ln\left(1+e^{-x}\right)=-2x$$

Apply the exponential function on both sides,

$$1+e^{-x}=e^{-2x}$$

Let's rewrite the exponential terms but with positive powers,

$$1+\frac{1}{e^x}=\frac{1}{e^{2x}}$$

Multiply both sides by e^{2x} to get rid of the denominators,

$$e^{2x}+e^x=1$$

Put all the terms on one side,

$$e^{2x}+e^x-1=0$$

Write it in the form $ax^2+bx+c=0$, to show that it is a quadratic equation in e^x,

$$\left(e^x\right)^2+e^x-1=0$$

Therefore, the final answer is,

$$\left(e^x\right)^2+e^x-1=0$$

(b) Hence solve the equation $\ln\left(1+e^{-x}\right)+2x=0$, giving your answer correct to 3 decimal places.

$$\left(e^x\right)^2+e^x-1=0$$

To solve the question, we simply have to solve the quadratic equation we derived in part (a),

$$\left(e^x\right)^2+e^x-1=0$$

Note: This is a hidden quadratic.

$$e^x=\frac{-1\pm\sqrt{1^2-4\times1\times-1}}{2(1)}$$

$$e^x=\frac{-1\pm\sqrt{5}}{2}$$

Let's split this into two equations,

$$e^x = \frac{-1-\sqrt{5}}{2} \qquad e^x = \frac{-1+\sqrt{5}}{2}$$

Get rid of the exponential function using the natural logarithm, $\ln$,

$$x = \ln\left(\frac{-1-\sqrt{5}}{2}\right) \qquad x = \ln\left(\frac{-1+\sqrt{5}}{2}\right)$$

$$x = \text{no solutions} \qquad x = -0.481$$

Note: Remember that $\ln$ only takes positive values.

Therefore, the final answer is,

$$x = -0.481$$

4. Solve the equation

$$\ln\left(1+e^{-3x}\right) = 2$$

Give the answer correct to 3 decimal places. (9709/32/O/N/20 number 1)

$$\ln\left(1+e^{-3x}\right) = 2$$

Apply the exponential function on both sides,

$$1+e^{-3x} = e^2$$

Subtract 1 from both sides,

$$e^{-3x} = e^2 - 1$$

Apply the natural logarithm on both sides,

$$-3x = \ln\left(e^2-1\right)$$

Divide both sides by -3,

$$x = -\frac{1}{3}\ln\left(e^2-1\right)$$

Give x correct to 3 decimal places,

$$x = -0.618$$

Therefore, the final answer is,

$$x = -0.618$$

5. The variables x and y satisfy the relation $2^y = 3^{1-2x}$. (9709/32/O/N/20 number 3)

   (a) By taking logarithms, show that the graph of y against x is a straight line. State the exact value of the gradient of this line.

$$2^y = 3^{1-2x}$$

Apply the natural logarithm to both sides,

$$\ln\left(2^y\right) = \ln\left(3^{1-2x}\right)$$

Use laws of logarithms to bring down the power,

$$y \ln 2 = (1 - 2x) \ln 3$$

Expand the bracket on the right-hand side,

$$y \ln 2 = \ln 3 - 2x \ln 3$$

$$y \ln 2 = -2 \ln 3x + \ln 3$$

Divide both sides by $\ln 2$ to make y the subject of the formula,

$$y = \frac{-2 \ln 3}{\ln 2} x + \frac{\ln 3}{\ln 2}$$

Notice how this is in the form $y = mx + c$,

$$y = \color{blue}{\frac{-2 \ln 3}{\ln 2}} x + \color{green}{\frac{\ln 3}{\ln 2}}$$

$$y = \color{blue}{m} x + \color{green}{c}$$

We can read off our gradient, m,

$$m = \frac{-2 \ln 3}{\ln 2}$$

Therefore, the final answer is,

$$m = \frac{-2 \ln 3}{\ln 2}$$

(b) Find the exact x-coordinate of the point of intersection of this line with the line $y = 3x$. Give your answer in the form $\frac{\ln a}{\ln b}$, where a and b are integers.

$$y = \frac{-2 \ln 3}{\ln 2} x + \frac{\ln 3}{\ln 2} \qquad y = 3x$$

To find a point of intersection, we have to solve simultaneously,

$$\frac{-2 \ln 3}{\ln 2} x + \frac{\ln 3}{\ln 2} = 3x$$

Put all the terms containing x on one side,

$$3x + \frac{2 \ln 3}{\ln 2} x = \frac{\ln 3}{\ln 2}$$

Factor out the x,

$$x \left(3 + \frac{2 \ln 3}{\ln 2}\right) = \frac{\ln 3}{\ln 2}$$

Let's simplify the bracket,

$$x\left(\frac{3\ln 2+2\ln 3}{\ln 2}\right)=\frac{\ln 3}{\ln 2}$$
$$x\left(\frac{\ln 2^3+\ln 3^2}{\ln 2}\right)=\frac{\ln 3}{\ln 2}$$
$$x\left(\frac{\ln 8+\ln 9}{\ln 2}\right)=\frac{\ln 3}{\ln 2}$$
$$x\left(\frac{\ln 8\times 9}{\ln 2}\right)=\frac{\ln 3}{\ln 2}$$
$$x\left(\frac{\ln 72}{\ln 2}\right)=\frac{\ln 3}{\ln 2}$$

Make x the subject of the formula,

$$x=\frac{\ln 3}{\ln 2}\times\frac{\ln 2}{\ln 72}$$
$$x=\frac{\ln 3}{\ln 72}$$

Therefore, the final answer is,

$$x=\frac{\ln 3}{\ln 72}$$

6. Solve the equation $\ln(x^3-3)=3\ln x-\ln 3$. Give your answer correct to 3 significant figures. (9709/32/F/M/21 number 1)

$$\ln\left(x^3-3\right)=3\ln x-\ln 3$$

Let's combine the natural logarithms on the right hand side,

$$\ln\left(x^3-3\right)=\ln x^3-\ln 3$$
$$\ln\left(x^3-3\right)=\ln\frac{x^3}{3}$$

Apply the exponential function to both sides,

$$x^3-3=\frac{x^3}{3}$$

Multiply through by 3 to get rid of the denominator,

$$3x^3-9=x^3$$

Put all the terms in x^3 on one side and solve for x,

$$3x^3-x^3=9$$
$$2x^3=9$$
$$x^3=\frac{9}{2}$$
$$x=\left(\frac{9}{2}\right)^{\frac{1}{3}}$$
$$x=1.65$$

Therefore, the final answer is,

$$x = 1.65$$

7. It is given that $x = \ln(2y - 3) - \ln(y + 4)$.

   Express y in terms of x. (9709/32/F/M/23 number 1)

$$x = \ln(2y - 3) - \ln(y + 4)$$

We need to make y the subject of the formula. Let's combine the natural logarithms on the right hand side using laws of logarithms,

$$x = \ln\left(\frac{2y - 3}{y + 4}\right)$$

Apply the exponential function on both sides,

$$e^x = \frac{2y - 3}{y + 4}$$

Get rid of the denominator,

$$e^x(y + 4) = 2y - 3$$

Expand the bracket,

$$ye^x + 4e^x = 2y - 3$$

Put all the terms containing y on one side,

$$2y - ye^x = 4e^x + 3$$

Factor out y,

$$y\left(2 - e^x\right) = 4e^x + 3$$

Make y the subject of the formula,

$$y = \frac{4e^x + 3}{2 - e^x}$$

Therefore, the final answer is,

$$y = \frac{4e^x + 3}{2 - e^x}$$

8. Solve the equation

$$3e^{2x} - 4e^{-2x} = 5$$

   Give the answer correct to 3 decimal places. (9709/31/M/J/23 number 1)

$$3e^{2x} - 4e^{-2x} = 5$$

Make all the powers positive,

$$3e^{2x} - \frac{4}{e^{2x}} = 5$$

Get rid of the denominator,

$$3e^{4x} - 4 = 5e^{2x}$$

Put all the terms on one side,

$$3e^{4x} - 5e^{2x} - 4 = 0$$

Notice how this is a hidden quadratic,

$$3\left(e^{2x}\right)^2 - 5e^{2x} - 4 = 0$$

Solve the hidden quadratic using the quadratic formula,

$$e^{2x} = \frac{5 \pm \sqrt{5^2 - 4 \times 3 \times -4}}{2(3)}$$

$$e^{2x} = \frac{5 \pm \sqrt{73}}{6}$$

Let's split this into two equations,

$$e^{2x} = \frac{5 - \sqrt{73}}{6} \qquad e^{2x} = \frac{5 + \sqrt{73}}{6}$$

Solve for x in both equations,

$$2x = \ln\left(\frac{5 - \sqrt{73}}{6}\right) \qquad 2x = \ln\left(\frac{5 + \sqrt{73}}{6}\right)$$

$$x = \text{no solutions} \qquad x = \frac{1}{2}\ln\left(\frac{5 + \sqrt{73}}{6}\right)$$

$$x = 0.407$$

Therefore, the final answer is,

$$x = 0.407$$

9. Solve the equation $\ln(2x^2 - 3) = 2\ln x - \ln 2$, giving your answer in an exact form. (9709/32/M/J/23 number 2)

$$\ln\left(2x^2 - 3\right) = 2\ln x - \ln 2$$

Let's combine the two logarithms on the right-hand side,

$$\ln\left(2x^2 - 3\right) = \ln x^2 - \ln 2$$

$$\ln\left(2x^2 - 3\right) = \ln\frac{x^2}{2}$$

Apply the exponential function on both sides,

$$2x^2 - 3 = \frac{x^2}{2}$$

Multiply through by 2 to get rid of the denominator,

$$4x^2 - 6 = x^2$$

Solve for x,

$$3x^2 = 6$$

$$x^2 = 2$$

$$x = \pm\sqrt{2}$$

Remember that the logarithmic function does not take negative values, so we disregard $-\sqrt{2}$,

$$x = \sqrt{2}$$

Therefore, the final answer is,

$$x = \sqrt{2}$$