Pure Maths 3

3.4 Differentiation - Easy

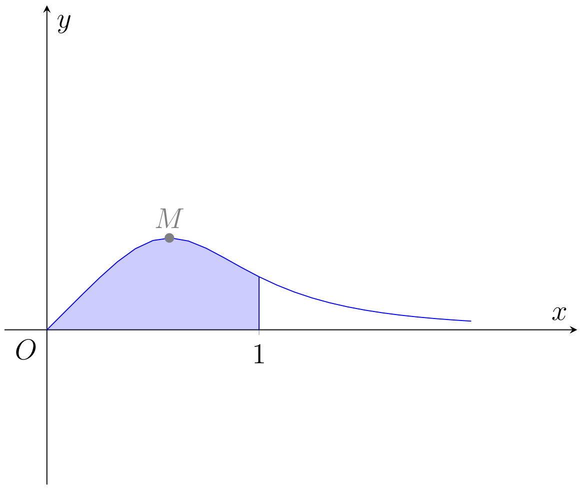
Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**A2 Level**
Component:	**Pure Mathematics 3**
Topic:	**3.4 Differentiation**
Difficulty:	**Easy**

Questions

1. The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leq x \leq \pi$. (9709/31/M/J/20 number 4)

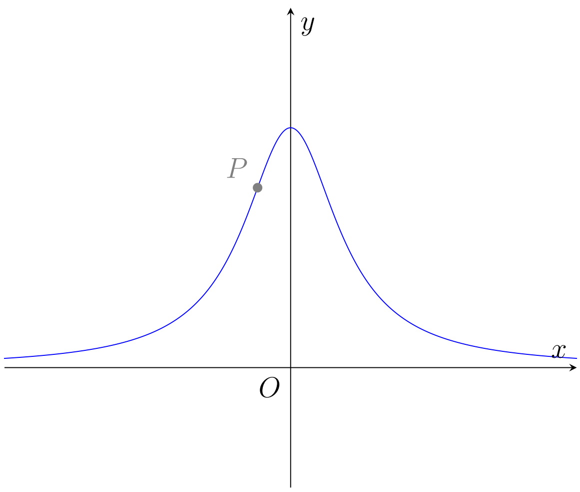
   (a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places.

   (b) Determine whether the stationary point is a maximum or a minimum.

2. 

   The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geq 0$, and its maximum point M. (9709/32/M/J/20 number 6a)

   Find the x-coordinate of M, giving your answer correct to 3 decimal places.

3. 

   The diagram shows the curve with parametric equations

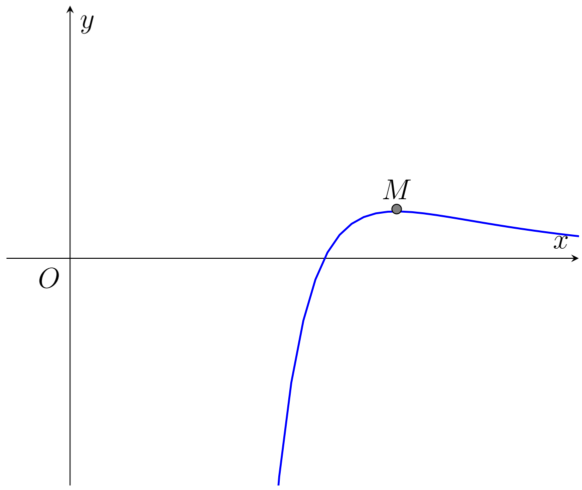
   $$x = \tan\theta \qquad y = \cos^2\theta$$

   for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. (9709/32/O/N/20 number 5)

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$.
The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P.

4.



The diagram shows the curve $y = \frac{\ln x}{x^4}$ and its maximum point M. (9709/33/M/J/21 number 8a)
Find the exact coordinates of M.

5. The curve with equation $y = xe^{1-2x}$ has one stationary point. (9709/31/O/N/21 number 3)

(a) Find the coordinates of this point.

(b) Determine whether the stationary point is a maximum or a minimum.

6. The equation of the curve is $\ln(x+y) = x - 2y$. (9709/33/O/N/21 number 7)

(a) Show that $\frac{dy}{dx} = \frac{x+y-1}{2(x+y)+1}$.

(b) Find the coordinates of the point on the curve where the tangent is parallel to the x-axis.

7. Let $f(x) = \frac{1}{(9-x)\sqrt{x}}$. (9709/33/O/N/21 number 9a)
Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$.

8. The curve $y = e^{-4x}\tan x$ has two stationary points in the interval $0 \le x < \frac{1}{2}\pi$. (9709/33/M/J/22 number 4)

(a) Obtain an expression for $\frac{dy}{dx}$ and show it can be written in the form $\sec^2 x(a + b\sin 2x)e^{-4x}$, where a and b are constants.

(b) Hence find the exact coordinates of the two stationary points.

9. The parametric equations of a curve are

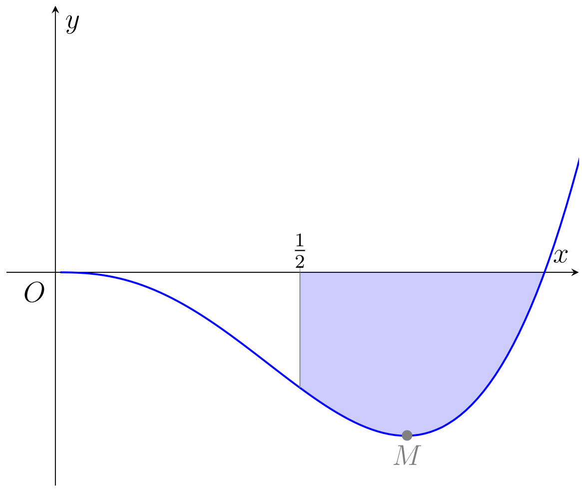
$$x = te^{2t}, \quad y = t^2 + t + 3$$

(9709/32/F/M/23 number 5)

(a) Show that $\frac{dy}{dx} = e^{-2t}$.

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(0, 3 - \frac{1}{e^4}\right)$.

10.



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M. (9709/32/F/M/23 number 8a)

Find the exact coordinates of M.

11. The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant. (9709/31/M/J/23 number 5)

   (a) Show that $\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$.

   (b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis.

Answers

1. The curve with equation $y = e^{2x}(\sin x + 3\cos x)$ has a stationary point in the interval $0 \leq x \leq \pi$. (9709/31/M/J/20 number 4)

   (a) Find the x-coordinate of this point, giving your answer correct to 2 decimal places.

$$y = e^{2x}(\sin x + 3\cos x)$$

Let's use the product rule to differentiate,

$$\frac{dy}{dx} = e^{2x}(\cos x - 3\sin x) + (\sin x + 3\cos x)2e^{2x}$$

Factor out e^{2x},

$$\frac{dy}{dx} = e^{2x}(\cos x - 3\sin x + 2(\sin x + 3\cos x))$$
$$\frac{dy}{dx} = e^{2x}(\cos x - 3\sin x + 2\sin x + 6\cos x)$$
$$\frac{dy}{dx} = e^{2x}(7\cos x - \sin x)$$

Since this is a stationary point, let's equate our first derivative to 0,

$$e^{2x}(7\cos x - \sin x) = 0$$

Now let's solve for x,

$$e^{2x} = 0 \qquad 7\cos x - \sin x = 0$$
$$2x = \ln 0$$
$$x = \text{no solutions}$$

$$7\cos x - \sin x = 0$$

Divide through by $\cos x$,

$$7 - \tan x = 0$$

Solve the trig equation,

$$\tan x = 7$$
$$x = \tan^{-1} 7$$
$$x = 1.43$$

Therefore, the final answer is,

$$x = 1.43$$

   (b) Determine whether the stationary point is a maximum or a minimum.

$$x = 1.43 \qquad \frac{dy}{dx} = e^{2x}(7\cos x - \sin x)$$

To determine the nature, let's pick a point either side of the stationary point,

$$\text{At } x = 1 \quad \text{At } x = 2$$

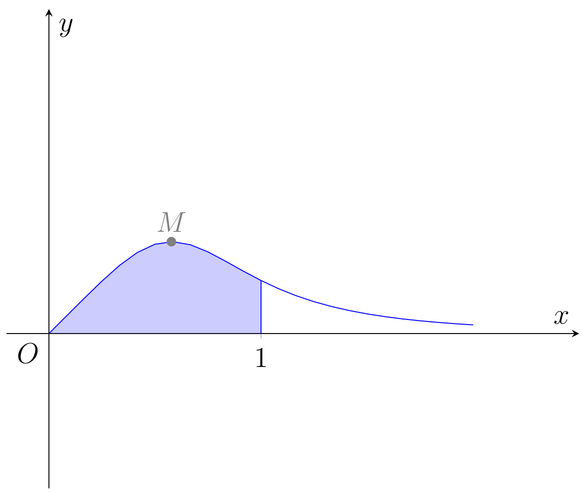
$$e^{2(1)}\left(7\cos 1 - \sin 1\right) \quad e^{2(2)}\left(7\cos 2 - \sin 2\right)$$
$$21.7 \quad -208.7$$

Note: If the gradient, $\frac{dy}{dx}$, is positive on the left side of a stationary point, it is maximum point. If the gradient is negative on the left side of a stationary point, it is minimum point.

Therefore, the final answer is,

The stationary point is a maximum point

2.



The diagram shows the curve $y = \frac{x}{1+3x^4}$, for $x \geq 0$, and its maximum point M. (9709/32/M/J/20 number 6a)

Find the x-coordinate of M, giving your answer correct to 3 decimal places.

$$y = \frac{x}{1+3x^4}$$

Let's use the quotient rule to find the first derivative,

$$\frac{dy}{dx} = \frac{\left(1+3x^4\right)(1) - x\left(12x^3\right)}{\left(1+3x^4\right)^2}$$

$$\frac{dy}{dx} = \frac{\left(1+3x^4\right) - x\left(12x^3\right)}{\left(1+3x^4\right)^2}$$

$$\frac{dy}{dx} = \frac{1+3x^4-12x^4}{\left(1+3x^4\right)^2}$$

$$\frac{dy}{dx} = \frac{1-9x^4}{\left(1+3x^4\right)^2}$$

Let's equate our first derivative to 0 to find the x-coordinate of M,

$$\frac{1-9x^4}{(1+3x^4)^2}=0$$

$$1-9x^4=0$$

$$9x^4=1$$

$$x^4=\frac{1}{9}$$

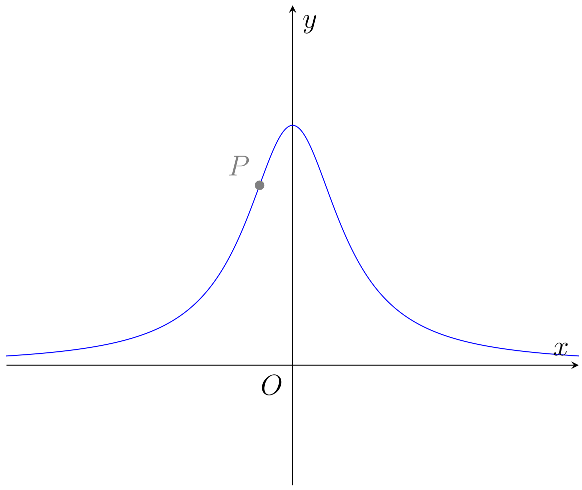
$$x=\left(\frac{1}{9}\right)^{\frac{1}{4}}$$

$$x=0.577$$

Therefore, the final answer is,

$$x=0.577$$

3.



The diagram shows the curve with parametric equations

$$x=\tan\theta \qquad y=\cos^2\theta$$

for $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$. (9709/32/O/N/20 number 5)

(a) Show that the gradient of the curve at the point with parameter θ is $-2\sin\theta\cos^3\theta$.

$$x=\tan\theta \qquad y=\cos^2\theta$$

Let's differentiate these two parametric equations,

$$\frac{dx}{d\theta}=\sec^2 x \qquad \frac{dy}{d\theta}=-2\cos\theta\sin\theta$$

Now let's create a chain rule for $\frac{dy}{dx}$,

$$\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

Substitute into the chain rule,

$$\frac{dy}{dx} = -2\cos\theta\sin\theta \times \frac{1}{\sec^2 x}$$

$$\frac{dy}{dx} = -2\cos\theta\sin\theta \times \frac{1}{\frac{1}{\cos^2 x}}$$

$$\frac{dy}{dx} = -2\cos\theta\sin\theta \times \cos^2 x$$

$$\frac{dy}{dx} = -2\sin\theta\cos^3\theta$$

Therefore, the final answer is,

$$\frac{dy}{dx} = -2\sin\theta\cos^3\theta$$

The gradient of the curve has its maximum value at the point P.

(b) Find the exact value of the x-coordinate of P.

$$\frac{dy}{dx} = -2\sin\theta\cos^3\theta$$

We need to find the derivative of the gradient function. Let's use the product rule,

$$\frac{d^2y}{dx^2} = -2\sin\theta\left(-3\cos^2\theta\sin\theta\right) + \left(\cos^3\theta\right) \times -2\cos\theta$$

$$\frac{d^2y}{dx^2} = 6\cos^2\theta\sin^2\theta - 2\cos^4\theta$$

Equate to 0**,**

$$6\cos^2\theta\sin^2\theta - 2\cos^4\theta = 0$$

Rewrite $\sin^2\theta$ **in terms of** $\cos^2\theta$ **to create a quadratic equation,**

$$6\cos^2\theta\left(1-\cos^2\theta\right) - 2\cos^4\theta = 0$$

$$6\cos^2\theta - 6\cos^4\theta - 2\cos^4\theta = 0$$

$$-8\cos^4\theta + 6\cos^2\theta = 0$$

Factor out $-2\cos^2\theta$**,**

$$-2\cos^2\theta\left(4\cos^2\theta - 3\right) = 0$$

$$-2\cos^2\theta = 0 \qquad 4\cos^2\theta - 3 = 0$$

$$\cos^2\theta = 0 \qquad \cos^2\theta = \frac{3}{4}$$

$$\cos\theta = 0 \qquad \cos\theta = \pm\sqrt{\frac{3}{4}}$$

$$\theta = \cos^{-1} 0$$

$$\theta = \frac{1}{2}\pi$$

$$\cos\theta = \pm\sqrt{\frac{3}{4}}$$

$$\theta = \cos^{-1}\left(-\sqrt{\frac{3}{4}}\right) \qquad \theta = \cos^{-1}\left(\sqrt{\frac{3}{4}}\right)$$

$$\theta = \frac{5}{6}\pi \qquad \theta = \frac{1}{6}\pi$$

Now let's find x,

$$x = \tan\theta$$

$$\text{At } \theta = \frac{5}{6}\pi \qquad \text{At } \theta = \frac{1}{6}\pi$$

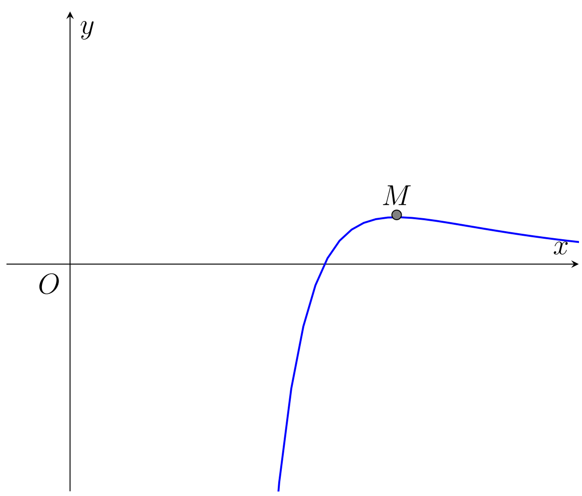
$$x = \tan\left(\frac{5}{6}\pi\right) \qquad x = \tan\left(\frac{1}{6}\pi\right)$$

$$x = -\frac{\sqrt{3}}{3} \qquad x = \frac{\sqrt{3}}{3}$$

From the diagram, we can tell that x is negative. Therefore, the final answer is,

$$x = -\frac{\sqrt{3}}{3}$$

4.



The diagram shows the curve $y = \frac{\ln x}{x^4}$ and its maximum point M. (9709/33/M/J/21 number 8a)
Find the exact coordinates of M.

$$y = \frac{\ln x}{x^4}$$

Use the quotient rule to find the first derivative,

$$\frac{dy}{dx} = \frac{x^4\left(\frac{1}{x}\right) - \ln x\,(4x^3)}{x^8}$$

$$\frac{dy}{dx} = \frac{x^3 - \ln x\,(4x^3)}{x^8}$$

Factor out x^3 in the numerator,

$$\frac{dy}{dx} = \frac{x^3(1-4\ln x)}{x^8}$$

$$\frac{dy}{dx} = \frac{1-4\ln x}{x^5}$$

Equate the first derivative to 0,

$$\frac{1-4\ln x}{x^5} = 0$$

Solve for x,

$$1-4\ln x = 0$$

$$4\ln x = 1$$

$$\ln x = \frac{1}{4}$$

$$x = e^{\frac{1}{4}}$$

Now let's evaluate y,

$$y = \frac{\ln x}{x^4}$$

$$y = \frac{\ln e^{\frac{1}{4}}}{\left(e^{\frac{1}{4}}\right)^4}$$

$$y = \frac{\frac{1}{4}}{e}$$

$$y = \frac{1}{4e}$$

Therefore, the final answer is,

$$\left(e^{\frac{1}{4}}, \frac{1}{4e}\right)$$

5. The curve with equation $y = xe^{1-2x}$ has one stationary point. (9709/31/O/N/21 number 3)

   (a) Find the coordinates of this point.

$$y = xe^{1-2x}$$

Let's use the product rule to find the first derivative,

$$\frac{dy}{dx} = x\left(-2e^{1-2x}\right) + e^{1-2x}(1)$$

$$\frac{dy}{dx} = -2xe^{1-2x} + e^{1-2x}$$

Factor out e^{1-2x},

$$\frac{dy}{dx} = e^{1-2x}(-2x+1)$$

Equate the first derivative to 0,

$$e^{1-2x}(-2x+1)=0$$

Solve for x,

$$e^{1-2x}=0 \qquad -2x+1=0$$
$$1-2x=\ln(0) \qquad 2x=1$$
$$x=\text{no solutions} \qquad x=\frac{1}{2}$$

Now let's evaluate y,

$$y=\frac{1}{2}e^{1-2\left(\frac{1}{2}\right)}$$
$$y=\frac{1}{2}e^{0}$$
$$y=\frac{1}{2}$$

Therefore, the final answer is,

$$\left(\frac{1}{2},\frac{1}{2}\right)$$

(b) Determine whether the stationary point is a maximum or a minimum.

$$x=\frac{1}{2} \qquad \frac{dy}{dx}=e^{1-2x}(-2x+1)$$

Let's pick a point either side of the gradient,

$$\text{At } x=0 \qquad \text{At } x=1$$
$$e^{1-2(0)}(-2(0)+1) \qquad e^{1-2(1)}(-2(1)+1)$$
$$2.718 \qquad -0.368$$

The gradient is positive on the left-hand side of the stationary point and negative on the right-hand side. This is a maximum point.

Therefore, the final answer is,

This is a maximum point.

6. The equation of the curve is $\ln(x+y)=x-2y$. (9709/33/O/N/21 number 7)

(a) Show that $\frac{dy}{dx}=\frac{x+y-1}{2(x+y)+1}$.

$$\ln(x+y)=x-2y$$

Let's differentiate our implicit function,

$$\frac{d}{dx}\left[\ln(x+y)=x-2y\right]$$
$$\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=1-2\frac{dy}{dx}$$
$$\frac{1}{x+y}+\frac{1}{x+y}\frac{dy}{dx}=1-2\frac{dy}{dx}$$

Put all the terms in $\frac{dy}{dx}$ on one side,

$$\frac{1}{x+y}\frac{dy}{dx}+2\frac{dy}{dx}=1-\frac{1}{x+y}$$
$$\frac{1}{x+y}\frac{dy}{dx}+2\frac{dy}{dx}=\frac{x+y-1}{x+y}$$

Factor out $\frac{dy}{dx}$,

$$\frac{dy}{dx}\left(\frac{1}{x+y}+2\right)=\frac{x+y-1}{x+y}$$
$$\frac{dy}{dx}\left(\frac{1+2(x+y)}{x+y}\right)=\frac{x+y-1}{x+y}$$
$$\frac{dy}{dx}\left(\frac{2(x+y)+1}{x+y}\right)=\frac{x+y-1}{x+y}$$

Make $\frac{dy}{dx}$ the subject of the formula,

$$\frac{dy}{dx}=\frac{x+y-1}{x+y}\times\frac{x+y}{2(x+y)+1}$$
$$\frac{dy}{dx}=\frac{x+y-1}{2(x+y)+1}$$

Therefore, the final answer is,

$$\frac{dy}{dx}=\frac{x+y-1}{2(x+y)+1}$$

(b) Find the coordinates of the point on the curve where the tangent is parallel to the x-axis.

$$\ln(x+y)=x-2y \quad \frac{dy}{dx}=\frac{x+y-1}{2(x+y)+1}$$

If the tangent is parallel to the x-axis it means that,

$$dy=0$$
$$x+y-1=0$$

This means we now have two equations in terms of x and y that we can solve simultaneously,

$$x+y-1=0 \qquad \ln(x+y)=x-2y$$
$$x=1-y$$

$$\ln(x+y)=x-2y$$
$$\ln(1-y+y)=1-y-2y$$
$$\ln(1)=1-3y$$
$$0=1-3y$$

$$3y = 1$$
$$y = \frac{1}{3}$$

$$x = 1 - \frac{1}{3}$$
$$x = \frac{2}{3}$$

Therefore, the final answer is,

$$\left(\frac{2}{3}, \frac{1}{3}\right)$$

7. Let $f(x) = \frac{1}{(9-x)\sqrt{x}}$. (9709/33/O/N/21 number 9a)

   Find the x-coordinate of the stationary point of the curve with equation $y = f(x)$.

$$f(x) = \frac{1}{(9-x)\sqrt{x}}$$

Let's use the quotient rule and product rule to find the first derivative,

$$f'(x) = \frac{(9-x)\sqrt{x}(0) - 1\left((9-x)\frac{1}{2}x^{-\frac{1}{2}} + \sqrt{x}(-1)\right)}{\left((9-x)\sqrt{x}\right)^2}$$

$$f'(x) = \frac{-\left((9-x)\frac{1}{2\sqrt{x}} - \sqrt{x}\right)}{(9-x)^2 x}$$

$$f'(x) = \frac{-(9-x)\frac{1}{2\sqrt{x}} + \sqrt{x}}{(9-x)^2 x}$$

Let's equate the first derivative to 0,

$$\frac{-(9-x)\frac{1}{2\sqrt{x}} + \sqrt{x}}{(9-x)^2 x} = 0$$

$$-(9-x)\frac{1}{2\sqrt{x}} + \sqrt{x} = 0$$

Multiply through by $2\sqrt{x}$ to get rid of the denominator,

$$-(9-x) + 2x = 0$$

$$-9 + x + 2x = 0$$

$$-9 + 3x = 0$$

$$3x = 9$$

$$x = 3$$

Therefore, the final answer is,

$$x = 3$$

8. The curve $y = e^{-4x}\tan x$ has two stationary points in the interval $0 \le x < \frac{1}{2}\pi$. (9709/33/M/J/22 number 4)

   (a) Obtain an expression for $\frac{dy}{dx}$ and show it can be written in the form $\sec^2 x(a + b\sin 2x)e^{-4x}$, where a and b are constants.

$$y = e^{-4x}\tan x$$

Let's use the product rule to find the first derivative,

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x\right) + \tan x\left(-4e^{-4x}\right)$$

Factor out e^{-4x},

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x - 4\tan x\right)$$

Let's rewrite $\tan x$ in terms of $\sin x$ and $\cos x$,

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x - 4\frac{\sin x}{\cos x}\right)$$

We need to find a way to change $\frac{\sin x}{\cos x}$ to $\sin 2x$. Let's play around with the double angle formula for $\sin$,

$$\sin 2x = 2\sin x\cos x$$

We need to make $\frac{\sin x}{\cos x}$ the subject of the formula. To do that, we will divide both sides by $\cos^2 x$,

$$\frac{\sin 2x}{\cos^2 x} = \frac{2\sin x\cos x}{\cos^2 x}$$

$$2\frac{\sin x}{\cos x} = \frac{\sin 2x}{\cos^2 x}$$

$$\frac{\sin x}{\cos x} = \frac{1}{2}\frac{\sin 2x}{\cos^2 x}$$

$$\frac{\sin x}{\cos x} = \frac{1}{2} \times \frac{1}{\cos^2 x} \times \sin 2x$$

$$\frac{\sin x}{\cos x} = \frac{1}{2} \times \sec^2 x \times \sin 2x$$

Let's substitute it into our first derivative,

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x - 4\frac{\sin x}{\cos x}\right)$$

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x - 4\left(\frac{1}{2}\sec^2 x\sin 2x\right)\right)$$

$$\frac{dy}{dx} = e^{-4x}\left(\sec^2 x - 2\sec^2 x\sin 2x\right)$$

Factor out $\sec^2 x$,

$$\frac{dy}{dx} = \sec^2 x\,(1 - 2\sin 2x)\,e^{-4x}$$

Therefore, the final answer is,

$$\frac{dy}{dx} = \sec^2 x\,(1 - 2\sin 2x)\,e^{-4x}$$

(b) Hence find the exact coordinates of the two stationary points.

$$\frac{dy}{dx} = \sec^2 x\,(1 - 2\sin 2x)\,e^{-4x}$$

Equate $\frac{dy}{dx}$ to 0,

$$\sec^2 x\,(1 - 2\sin 2x)\,e^{-4x} = 0$$
$$\sec^2 x = 0 \quad 1 - 2\sin 2x = 0 \quad e^{-4x} = 0$$
$$\sec x = 0 \quad 2\sin 2x = 1 \quad -4x = \ln 0$$
$$\frac{1}{\cos x} = 0 \quad \sin 2x = \frac{1}{2} \quad x = \text{no solutions}$$
$$\cos x = \frac{1}{0}$$
$$x = \text{no solutions}$$

$$\sin 2x = \frac{1}{2}$$
$$2x = \sin^{-1}\frac{1}{2}$$
$$P.V = \frac{1}{6}\pi$$
$$-\frac{1}{6}\pi + \pi = \frac{5}{6}\pi$$
$$2x = \frac{1}{6}\pi \quad 2x = \frac{5}{6}\pi$$
$$x = \frac{1}{12}\pi \quad x = \frac{5}{12}\pi$$

Therefore, the final answer is,

$$x = \frac{1}{12}\pi, \frac{5}{12}\pi$$

9. The parametric equations of a curve are

$$x = te^{2t}, \quad y = t^2 + t + 3$$

(9709/32/F/M/23 number 5)

(a) Show that $\frac{dy}{dx} = e^{-2t}$.

$$x = te^{2t}, \quad y = t^2 + t + 3$$

Let's start by differentiating the equations above,

$$\frac{dx}{dt} = t\left(2e^{2t}\right) + e^{2t}(1) \qquad \frac{dy}{dt} = 2t + 1$$

$$\frac{dx}{dt} = e^{2t}(2t+1) \qquad \frac{dy}{dt} = 2t + 1$$

Let's create a chain rule for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

Substitute into the chain rule,

$$\frac{dy}{dx} = 2t + 1 \times \frac{1}{e^{2t}(2t+1)}$$

$$\frac{dy}{dx} = \frac{1}{e^{2t}}$$

$$\frac{dy}{dx} = e^{-2t}$$

Therefore, the final answer is,

$$\frac{dy}{dx} = e^{-2t}$$

(b) Hence show that the normal to the curve, where $t = -1$, passes through the point $\left(0, 3 - \frac{1}{e^4}\right)$.

$$\frac{dy}{dx} = e^{-2t}$$

We need to find the equation of the normal. Let's start by finding the gradient of the tangent at $t = -1$,

$$\frac{dy}{dx} = e^{-2(-1)}$$

$$\frac{dy}{dx} = e^{2}$$

This means that the gradient of the normal is,

$$m = -\frac{1}{e^2}$$

Now let's find the set of coordinates where $t = -1$,

$$x = te^{2t}, \qquad y = t^2 + t + 3$$

$$x = (-1)e^{2(-1)}, \qquad y = (-1)^2 - 1 + 3$$

$$x = -e^{-2} \qquad y = 3$$

$$x = -\frac{1}{e^2} \qquad y = 3$$

Now let's find the equation of the normal,

$$y = mx + c \quad m = -\frac{1}{e^2} \quad \text{passing through } \left(-\frac{1}{e^2}, 3\right)$$

$$3 = -\frac{1}{e^2} \times -\frac{1}{e^2} + c$$
$$3 = \frac{1}{e^4} + c$$
$$c = 3 - \frac{1}{e^4}$$

This means that the equation of the normal is,

$$y = -\frac{1}{e^2}x + 3 - \frac{1}{e^4}$$

Let's substitute x with 0,

$$y = -\frac{1}{e^2}(0) + 3 - \frac{1}{e^4}$$
$$y = 3 - \frac{1}{e^4}$$

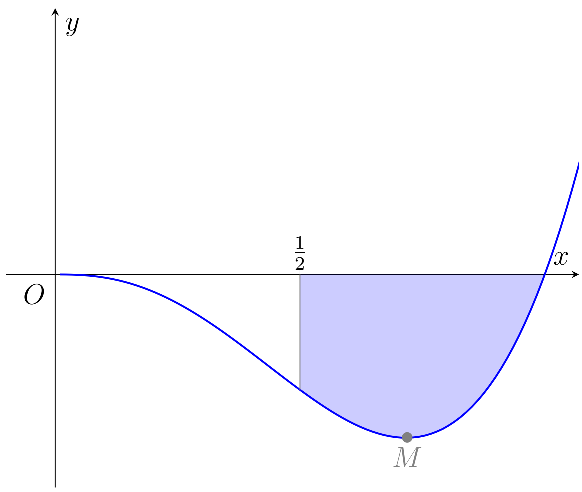
This satisfies the coordinates,

$$\left(0, 3 - \frac{1}{e^4}\right)$$

Therefore, the final answer is,

The normal to the curve, where $t = -1$, passes through the point $\left(0, 3 - \frac{1}{e^4}\right)$.

10.



The diagram shows the curve $y = x^3 \ln x$, for $x > 0$, and its minimum point M. (9709/32/F/M/23 number 8a)

Find the exact coordinates of M.

$$y = x^3 \ln x$$

Let's use the product rule to find the first derivative,

$$\frac{dy}{dx} = x^3\left(\frac{1}{x}\right) + \ln x\left(3x^2\right)$$

$$\frac{dy}{dx} = x^2 + 3\ln x x^2$$

Factor out x^2,

$$\frac{dy}{dx} = x^2\left(1 + 3\ln x\right)$$

Let's equate our first derivative to 0,

$$x^2\left(1 + 3\ln x\right) = 0$$

Solve for x,

$$x^2 = 0 \quad 1 + 3\ln x = 0$$

$$x = 0$$

$$1 + 3\ln x = 0$$

$$3\ln x = -1$$

$$\ln x = -\frac{1}{3}$$

$$x = e^{-\frac{1}{3}}$$

We are told that x is greater than 0,

$$x = e^{-\frac{1}{3}}$$

Now let's evaluate y,

$$y = x^3\ln x$$

$$y = \left(e^{-\frac{1}{3}}\right)^3 \ln e^{-\frac{1}{3}}$$

$$y = e^{-1} \times -\frac{1}{3}$$

$$y = -\frac{1}{3}e^{-1}$$

Therefore, the final answer is,

$$\left(e^{-\frac{1}{3}}, -\frac{1}{3}e^{-1}\right)$$

11. The equation of a curve is $x^2y - ay^2 = 4a^3$, where a is a non-zero constant. (9709/31/M/J/23 number 5)

    (a) Show that $\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$.

$$x^2y - ay^2 = 4a^3$$

Let's differentiate the implicit function, using the product rule where appropriate,

$$\frac{d}{dx}\left[x^2y - ay^2 = 4a^3\right]$$

$$x^2\frac{dy}{dx} + y(2x) - 2ay\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 2xy - 2ay\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} - 2ay\frac{dy}{dx} = -2xy$$

Make $\frac{dy}{dx}$ the subject of the formula,

$$\frac{dy}{dx}\left(x^2 - 2ay\right) = -2xy$$

$$\frac{dy}{dx} = \frac{-2xy}{x^2 - 2ay}$$

Factor out a negative sign in the denominator,

$$\frac{dy}{dx} = \frac{-2xy}{-\left(-x^2 + 2ay\right)}$$

Cancel out the negative signs in the numerator and denominator,

$$\frac{dy}{dx} = \frac{2xy}{-x^2 + 2ay}$$

$$\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$$

Therefore, the final answer is,

$$\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$$

(b) Hence find the coordinates of the points where the tangent to the curve is parallel to the y-axis.

$$x^2y - ay^2 = 4a^3 \qquad \frac{dy}{dx} = \frac{2xy}{2ay - x^2}$$

If the tangent is parallel to the y-axis then,

$$dx = 0$$

$$2ay - x^2 = 0$$

Now we have two equations that we can solve simultaneously,

$$2ay - x^2 = 0 \qquad x^2y - ay^2 = 4a^3$$

$$x^2 = 2ay$$

$$\begin{aligned}(2ay)y - ay^2 &= 4a^3\\ 2ay^2 - ay^2 &= 4a^3\\ ay^2 &= 4a^3\\ y^2 &= \frac{4a^3}{a}\\ y^2 &= 4a^2\\ y &= \pm\sqrt{4a^2}\\ y &= \pm 2a\end{aligned}$$

Now let's find the x-coordinates,

$$x^2 = 2ay$$

$$\begin{array}{rr}\text{At } y = -2a & \text{At } y = 2a\\ x^2 = 2a(-2a) & x^2 = 2a(2a)\\ x^2 = -4a^2 & x^2 = 4a^2\\ x = \pm\sqrt{-4a^2} & x = \pm\sqrt{4a^2}\\ x = \text{No Solutions} & x = \pm 2a\end{array}$$

Note: The square root of a negative number has no real solutions.

Therefore, the final answer is,

$$(-2a, 2a),\ (2a, 2a)$$