Pure Maths 3

3.6 Numerical Solution of Equations - Easy

GCEALevelMaths9709

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**A2 Level**
Component:	**Pure Mathematics 3**
Topic:	**3.6 Numerical Solution of Equations**
Difficulty:	**Easy**

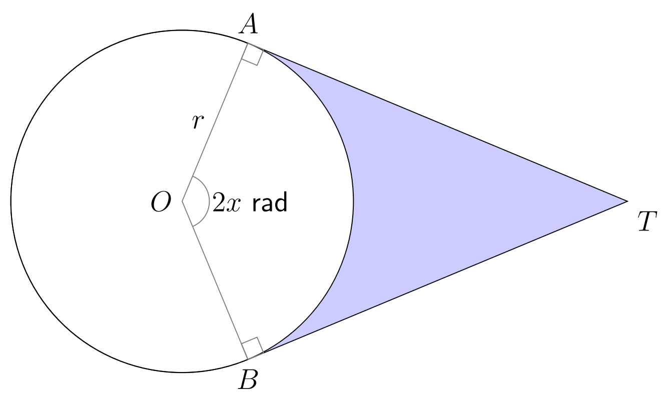
Questions

1. (a) By sketching a suitable pair of graphs, show that the equation $\sec x = 2 - \frac{1}{2}x$ has exactly one root in the interval $0 \leq x \leq \frac{1}{2}\pi$. (9709/32/F/M/20 number 3)

   (b) Verify by calculation that this root lies between 0.8 and 1.

   (c) Use the iterative formula $x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

2.



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle. (9709/31/M/J/20 number 6)

   (a) Show that x satisfies the equation $\tan x = \pi + x$.

   (b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4.

   (c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

   to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

3. (a) By sketching a pair of suitable graphs, show that the equation $\csc x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. (9709/31/O/N/20 number 5)

   (b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$$

   with initial value $x_1 = 2$, converges to one of these roots.

   Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

4. Let $f(x) = \frac{e^{2x}+1}{e^{2x}-1}$, for $x > 0$. (9709/32/F/M/21 number 9ab)

   (a) The equation $x = f(x)$ has one root, denoted by a.
   Verify by calculation that a lies between 1 and 1.5.

   (b) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

5. (a) By sketching a pair of suitable graphs show that the equation $x^5 = 2 + x$ has exactly one real root. (9709/33/M/J/20 number 6)

   (b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

   converges, then it converges to the root of the equation in part (a).

   (c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

6. The equation $\cot \frac{1}{2}x = 3x$ has one real root in the interval $0 < x < \pi$, denoted by α. (9709/32/M/J/23 number 6)

   (a) Show by calculation that α lies between 0.5 and 1.

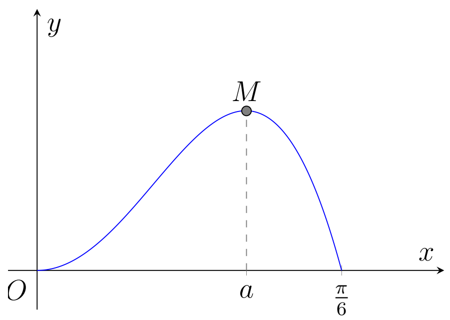
   (b) Show that, if a sequence of of positive values by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

   converges, then it converges to α.

   (c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

7.



The diagram shows part of the curve $y = x^2 \cos 3x$ for $0 \le x \le \frac{1}{6}\pi$, and its maximum point M, where $x = a$. (9709/33/M/J/23 number 5)

   (a) Show that a satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$.

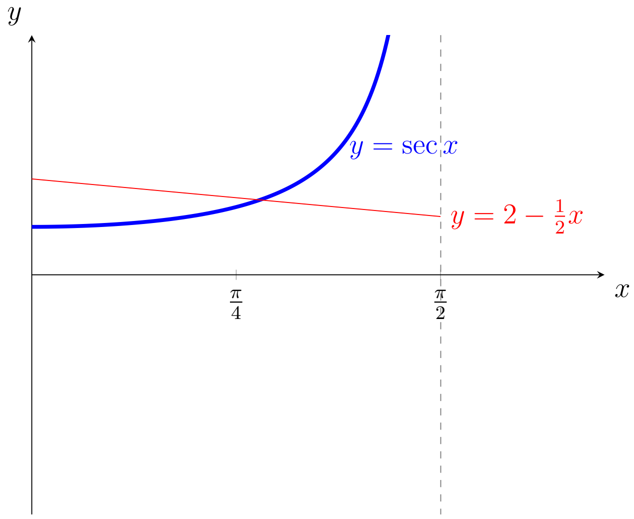
   (b) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

Answers

1. (a) By sketching a suitable pair of graphs, show that the equation $\sec x = 2 - \frac{1}{2}x$ has exactly one root in the interval $0 \leq x \leq \frac{1}{2}\pi$. (9709/32/F/M/20 number 3)

$$\sec x = 2 - \frac{1}{2}x$$

Sketch the graph of $y = \sec x$ and $y = 2 - \frac{1}{2}x$ on the same plane,



(b) Verify by calculation that this root lies between 0.8 and 1.

$$\sec x = 2 - \frac{1}{2}x$$

Put all the terms on one side, and the define it as $f(x)$,

$$f(x) = \sec x + \frac{1}{2}x - 2$$

Evaluate $f(0.8)$,

$$f(0.8) = \sec 0.8 + \frac{1}{2}(0.8) - 2$$

Note: Remember that $\sec x = \frac{1}{\cos x}$ when you simplify using your calculator.

$$f(0.8) = -0.165$$

Evaluate $f(1)$,

$$f(1) = \sec 1 + \frac{1}{2}(1) - 2$$

$$f(1) = 0.351$$

Therefore, the final answer is,

Since there is a change in sign, a root lies between 0.8 and 1.

(c) Use the iterative formula $x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x_{n+1} = \cos^{-1}\left(\frac{2}{4-x_n}\right)$$

Let $x_0 = 1$,

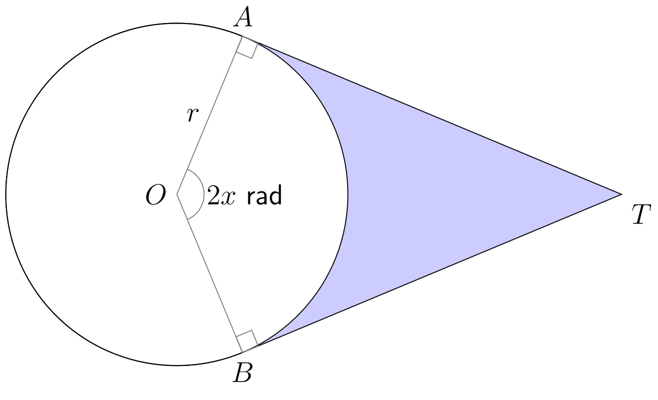
$$x_1 = \cos^{-1}\left(\frac{2}{4-1}\right) = 0.8411$$
$$x_2 = \cos^{-1}\left(\frac{2}{4-Ans}\right) = 0.8852$$
$$x_3 = 0.8736$$
$$x_4 = 0.8767 \approx 0.88$$
$$x_5 = 0.8758 \approx 0.88$$

Note: Instead of punching the result of the previous iteration into the current iteration, you can use the the Ans button on your calculator. Replace any x's in your iterative formula with 'Ans'. By simply pressing the equals button you get the result for you iteration. Keep pressing the equals button to get the results for the following iterations.

Therefore, the final answer is,

$$0.88$$

2.



The diagram shows a circle with centre O and radius r. The tangents to the circle at the points A and B meet at T, and angle AOB is $2x$ radians. The shaded region is bounded by the tangents AT and BT, and by the minor arc AB. The area of the shaded region is equal to the area of the circle. (9709/31/M/J/20 number 6)

(a) Show that x satisfies the equation $\tan x = \pi + x$.

$$\text{Area of shaded region} = \text{Area of circle}$$

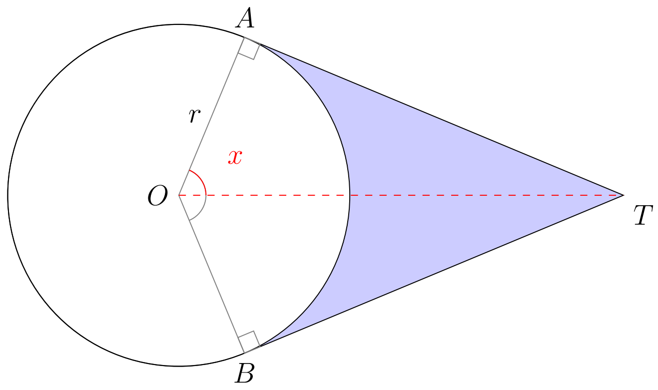
The area of the circle can be found by,

$$\pi r^2$$

Now let's find the area of the shaded region,

$$\text{Area of shaded region} = \text{Area of kite AOBT} - \text{Area of sector AOB}$$

We need to find the area of the kite. Let's split it into two triangles,



Using $SOHCAHTOA$ we can deduce AT,

$$AT = r \tan x$$

Now let's find the area of the kite,

$$\text{Area of kite } AOBT = OA \times AT$$

$$\text{Area of kite } AOBT = r \times r \tan x$$

$$\text{Area of kite } AOBT = r^2 \tan x$$

Now let's find the area of sector AOB,

$$\frac{1}{2} r^2 (2x)$$

$$r^2 x$$

Now let's go back to our expression for the area of the shaded region,

$$\text{Area of shaded region} = r^2 \tan x - r^2 x$$

Factor out r^2,

$$\text{Area of shaded region} = r^2(\tan x - x)$$

We were told that the area of the shaded region is equal to the area of the circle,

$$\pi r^2 = r^2(\tan x - x)$$

Put all the terms one side,

$$\pi r^2 - r^2(\tan x - x) = 0$$

Factor out r^2,

$$r^2(\pi - \tan x + x) = 0$$

$$\pi - \tan x + x = 0$$

Make $\tan x$ the subject of the formula,

$$\tan x = \pi + x$$

Therefore, the final answer is,

$$\tan x = \pi + x$$

(b) This equation has one root in the interval $0 < x < \frac{1}{2}\pi$. Verify by calculation that this root lies between 1 and 1.4.

$$\tan x = \pi + x$$

Put all the terms on one side and define this function as $f(x)$,

$$f(x) = \tan x - \pi - x$$

Evaluate $f(1)$,

$$f(1) = \tan 1 - \pi - 1$$

$$f(1) = -2.584$$

Evaluate $f(1.4)$,

$$f(1.4) = \tan 1.4 - \pi - 1.4$$

$$f(1.4) = 1.256$$

Therefore, the final answer is,

Since there is a change in sign, a root lies between 1 and 1.4.

(c) Use the iterative formula

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x_{n+1} = \tan^{-1}(\pi + x_n)$$

Let $x_0 = 1$**,**

$$x_1 = \tan^{-1}(\pi + 1) = 1.3339$$
$$x_2 = \tan^{-1}(\pi + Ans) = 1.3510 \approx 1.35$$
$$x_3 = 1.3518 \approx 1.35$$

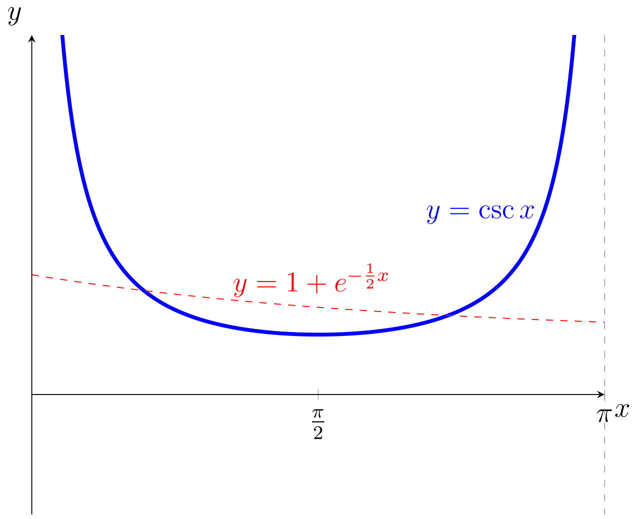
Therefore, the final answer is,

$$1.35$$

3. (a) By sketching a pair of suitable graphs, show that the equation $\csc x = 1 + e^{-\frac{1}{2}x}$ has exactly two roots in the interval $0 < x < \pi$. (9709/31/O/N/20 number 5)

$$\csc x = 1 + e^{-\frac{1}{2}x}$$

Sketch the graph of $y = \csc x$ **and the graph of** $y = 1 + e^{-\frac{1}{2}x}$ **on the same plane,**



(b) The sequence of values given by the iterative formula

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$$

with initial value $x_1 = 2$, converges to one of these roots.
Use the formula to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x_{n+1} = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}x_n} + 1}\right)$$

Let's evaluate x_2**,**

$$x_2 = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}(2)} + 1}\right) = 2.3217$$
$$x_3 = \pi - \sin^{-1}\left(\frac{1}{e^{-\frac{1}{2}(Ans)} + 1}\right) = 2.2760 \approx 2.28$$
$$x_4 = 2.2824 \approx 2.28$$

Therefore, the final answer is,

$$2.28$$

4. Let $f(x) = \frac{e^{2x}+1}{e^{2x}-1}$, for $x > 0$. (9709/32/F/M/21 number 9ab)

   (a) The equation $x = f(x)$ has one root, denoted by a.
   Verify by calculation that a lies between 1 and 1.5.

$$x = f(x)$$

Put all the terms on one side and we will call this function $g(x)$,

$$g(x) = f(x) - x$$

$$g(x) = \frac{e^{2x}+1}{e^{2x}-1} - x$$

Let's evaluate $g(1)$,

$$g(1) = \frac{e^{2(1)}+1}{e^{2(1)}-1} - 1$$
$$g(1) = 0.313$$

Let's evaluate $g(1.5)$,

$$g(1.5) = \frac{e^{2(1.5)}+1}{e^{2(1.5)}-1} - 1.5$$
$$g(1.5) = -0.395$$

Therefore, the final answer is,

Since there is a change in sign, a lies between 1 and 1.5.

   (b) Use an iterative formula based on the equation in part (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x = f(x)$$

$$x_{n+1} = f(x_n)$$
$$x_{n+1} = \frac{e^{2x_n}+1}{e^{2x_n}-1}$$

Let $x_0 = 1$,

$$x_1 = \frac{e^{2(1)}+1}{e^{2(1)}-1} = 1.3130$$
$$x_2 = \frac{e^{2(Ans)}+1}{e^{2(Ans)}-1} = 1.1560$$
$$x_3 = 1.2199$$
$$x_4 = 1.1910$$
$$x_5 = 1.2035 \approx 1.20$$
$$x_6 = 1.1980 \approx 1.20$$

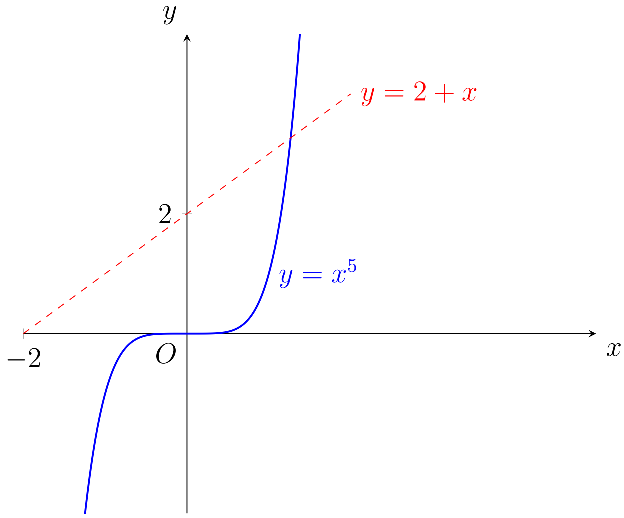
Therefore, the final answer is,

$$a = 1.20$$

5. (a) By sketching a pair of suitable graphs show that the equation $x^5 = 2 + x$ has exactly one real root. (9709/33/M/J/20 number 6)

$$x^5 = 2 + x$$

Sketch the graph of $y = x^5$ and the graph of $y = 2 + x$ on the same plane,



(b) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

converges, then it converges to the root of the equation in part (a).

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

This means that we have to show that the iterative formula simplifies to give the equation in part (a),

$$x = \frac{4x^5 + 2}{5x^4 - 1}$$

Get rid of the denominator,

$$x\left(5x^4 - 1\right) = 4x^5 + 2$$

Expand the bracket,

$$5x^5 - x = 4x^5 + 2$$

Make x^5 the subject of the formula,

$$5x^5 - 4x^5 = 2 + x$$

$$x^5 = 2 + x$$

Therefore, the final answer is,

The sequence converges to the root of the equation in part (a)

(c) Use the iterative formula with initial value $x_1 = 1.5$ to calculate the root correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

$$x_{n+1} = \frac{4x_n^5 + 2}{5x_n^4 - 1}$$

Let's evaluate x_2,

$$x_2 = \frac{4(1.5)^5 + 2}{5(1.5)^4 - 1} = 1.33162$$
$$x_3 = \frac{4(Ans)^5 + 2}{5(Ans)^4 - 1} = 1.27352$$
$$x_4 = 1.26724 \approx 1.267$$
$$x_5 = 1.26717 \approx 1.267$$

Therefore, the final answer is,

$$1.267$$

6. The equation $\cot \frac{1}{2}x = 3x$ has one real root in the interval $0 < x < \pi$, denoted by α. (9709/32/M/J/23 number 6)

(a) Show by calculation that α lies between 0.5 and 1.

$$\cot \frac{1}{2}x = 3x$$

Put all the terms on one side and define this as $f(x)$,

$$f(x) = \cot \frac{1}{2}x - 3x$$

Evaluate $f(0.5)$,

$$f(0.5) = \cot \frac{1}{2}(0.5) - 3(0.5)$$

Note: Remember that $\cot$ is $\frac{1}{\tan}$ when using your calculator.

$$f(0.5) = 2.416$$

Evaluate $f(1)$,

$$f(1) = \cot \frac{1}{2}(1) - 3(1)$$
$$f(1) = -1.170$$

Therefore, the final answer is,

Since there is a change in sign, a root lies between 0.5 and 1.

(b) Show that, if a sequence of of positive values by the iterative formula

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

converges, then it converges to α.

$$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right)$$

This means we have to show that the iterative formula simplifies to give the equation in the stem of the question, $\cot\frac{1}{2}x = 3x$**,**

$$x = \frac{1}{3}\left(x + 4\tan^{-1}\left(\frac{1}{3x}\right)\right)$$

Multiply both sides by 3**,**

$$3x = x + 4\tan^{-1}\left(\frac{1}{3x}\right)$$

Subtract x **from both sides,**

$$2x = 4\tan^{-1}\left(\frac{1}{3x}\right)$$

Divide both sides by 4**,**

$$\frac{1}{2}x = \tan^{-1}\left(\frac{1}{3x}\right)$$

Apply the $\tan$ **function on both sides,**

$$\tan\frac{1}{2}x = \frac{1}{3}x$$

Make $3x$ **the subject of the formula,**

$$3x = \frac{1}{\tan\frac{1}{2}x}$$

$$3x = \cot\frac{1}{2}x$$

$$\cot\frac{1}{2}x = 3x$$

Therefore, the final answer is,

The sequence converges to α.

(c) Use this iterative formula to calculate α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x_{n+1} = \frac{1}{3}\left(x_n + 4\tan^{-1}\left(\frac{1}{3x_n}\right)\right)$$

Let $x_0 = 1$,

$$x_1 = \frac{1}{3}\left((1) + 4\tan^{-1}\left(\frac{1}{(1)}\right)\right) = 0.7623$$

$$x_2 = \frac{1}{3}\left((Ans) + 4\tan^{-1}\left(\frac{1}{(Ans)}\right)\right) = 0.8037$$

$$x_3 = 0.7921$$

$$x_4 = 0.7951$$

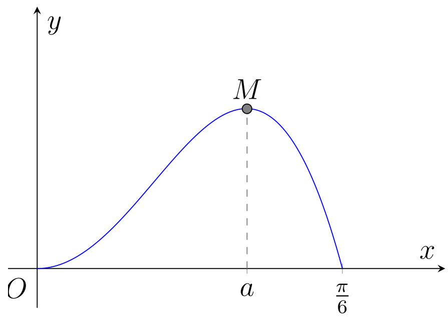
$$x_5 = 0.7943 \approx 0.79$$

$$x_6 = 0.7945 \approx 0.79$$

Therefore, the final answer is,

$$\alpha = 0.79$$

7.



The diagram shows part of the curve $y = x^2 \cos 3x$ for $0 \leq x \leq \frac{1}{6}\pi$, and its maximum point M, where $x = a$. (9709/33/M/J/23 number 5)

(a) Show that a satisfies the equation $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$.

$$y = x^2 \cos 3x$$

a is the x-coordinate at a stationary point. Let's find the first derivative of y using the product rule,

$$\frac{dy}{dx} = x^2 \times -3\sin 3x + \cos 3x \times 2x$$

$$\frac{dy}{dx} = -3x^2 \sin 3x + 2x \cos 3x$$

At a stationary point $\frac{dy}{dx} = 0$,

$$-3x^2 \sin 3x + 2x \cos 3x = 0$$

We are told that at this stationary point $x = a$,

$$-3a^2 \sin 3a + 2a \cos 3a = 0$$

Now we have to show that this equation simplifies to give $a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$,

$$-3a^2 \sin 3a + 2a \cos 3a = 0$$

Divide through by $\cos 3a$,

$$-3a^2 \tan 3a + 2a = 0$$

Factor out $3a^2$,

$$3a^2 \left(-\tan 3a + \frac{2}{3a} \right) = 0$$

$$-\tan 3a + \frac{2}{3a} = 0$$

Now the goal is to make a the subject of the formula. Add $\tan 3a$ to both sides,

$$\tan 3a = \frac{2}{3a}$$

Apply the $\tan^{-1}$ to both sides,

$$3a = \tan^{-1} \left(\frac{2}{3a} \right)$$

Divide both sides by 3,

$$a = \frac{1}{3} \tan^{-1} \left(\frac{2}{3a} \right)$$

Therefore, the final answer is,

$$a = \frac{1}{3} \tan^{-1} \left(\frac{2}{3a} \right)$$

(b) Use an iterative formula based on the equation in (a) to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$a_{n+1} = \frac{1}{3} \tan^{-1} \left(\frac{2}{3a_n} \right)$$

Let $a_0 = 0.5$,

$$a_1 = \frac{1}{3} \tan^{-1} \left(\frac{2}{3(0.5)} \right) = 0.3091$$

$$a_2 = \frac{1}{3} \tan^{-1} \left(\frac{2}{3(Ans)} \right) = 0.3789$$

$$a_3 = 0.3513$$

$$a_4 = 0.3619 \approx 0.36$$

$$a_5 = 0.3578 \approx 0.36$$

Therefore, the final answer is,

$$a = 0.36$$