Pure Maths 3

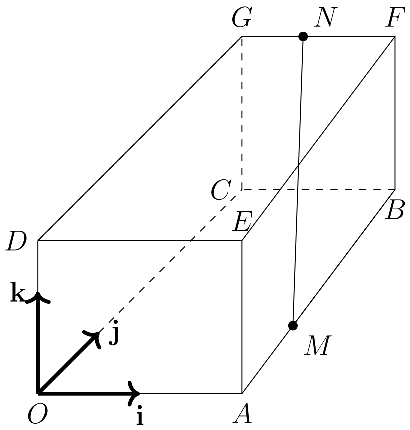
3.7 Vectors - Easy

GCEALevelMaths9709

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**A2 Level**
Component:	**Pure Mathematics 3**
Topic:	**3.7 Vectors**
Difficulty:	**Easy**

Questions

1.



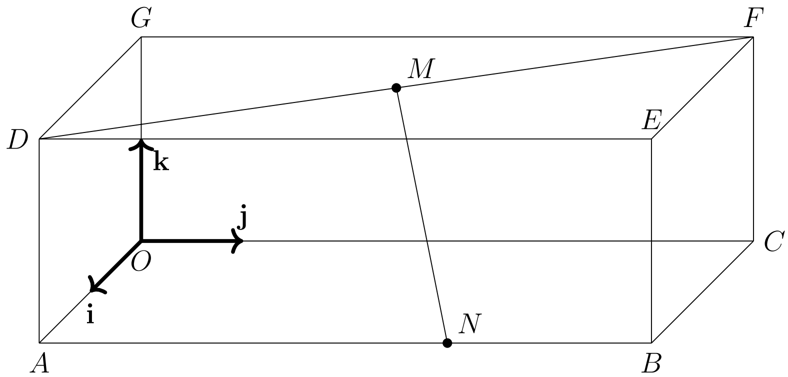
In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 3$ units and $OD = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OC and OD respectively. The point M on AB is such that $MB = 2AM$. The midpoint of FG is N. (9709/32/F/M/20 number 8)

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**.

(b) Find a vector equation for the line through M and N.

(c) Find the position vector of P, the foot of the perpendicular from D to the line through M and N.

2.



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OC and OG respectively. The point M is the midpoint of DF. The point N on AB is such that $AN = 3NB$. (9709/31/M/J/22 number 9)

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**.

(b) Find a vector equation for the line through M and N.

(c) Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$.

3. The lines l and m have vector equations

$$r = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad r = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

respectively, where a and b are constants. (9709/32/M/J/22 number 9)

(a) Given that l and m intersect, show that $2b - a = 4$.

(b) Given also that l and m are perpendicular, find the values of a and b.

(c) When a and b have these values, find the position vector of the point of intersection of l and m.

4. With respect to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. The line l has vector equation $r = 4\mathbf{i} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$. (9709/33/M/J/22 number 9)

   (a) Find in degrees the acute angle between the directions of OA and l.

   (b) Find the position vector of the foot of the perpendicular from A to l.

   (c) Hence find the position vector of the reflection of A in l.

5. With respect to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$$

   The midpoint of AC is M and the point N lies on BC, between B and C, and is such that $BN = 2NC$. (9709/33/O/N/22 number 9)

   (a) Find the position vectors of M and N.

   (b) Find a vector equation for the line through M and N.

   (c) Find the position vector of the point Q where the line through M and N intersects the line through A and B.

6. With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}$$

   (9709/32/F/M/23 number 10)

   (a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.
   The line l passes through the points A and B.

   (b) Find a vector equation for the line l.

   (c) Find the position vector of the point of intersection of the line l and the line passing through C and D.

7. Relative to the origin, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$$

   The quadrilateral $ABCD$ is a parallelogram. (9709/31/M/J/23 number 6)

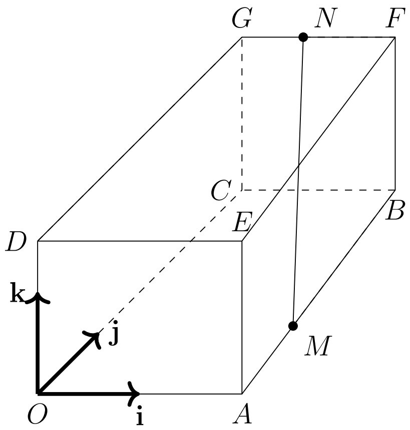
   (a) Find the position vector of D.

   (b) The angle between BA and BC is θ.
   Find the exact value of $\cos\theta$.

   (c) Hence find the area of $ABCD$, giving your answers in the form $p\sqrt{q}$, where p and q are integers.

Answers

1.



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 3$ units and $OD = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OC and OD respectively. The point M on AB is such that $MB = 2AM$. The midpoint of FG is N. (9709/32/F/M/20 number 8)

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**.

We can write $\overrightarrow{OM}$ as,

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$
$$\overrightarrow{OM} = 2\mathbf{i} + \overrightarrow{AM}$$

AM is a third of AB,

$$\overrightarrow{AM} = \frac{1}{3} \times 3\mathbf{j}$$
$$\overrightarrow{AM} = \mathbf{j}$$

Now let's evaluate $\overrightarrow{OM}$,

$$\overrightarrow{OM} = 2\mathbf{i} + j$$

We can write $\overrightarrow{MN}$ as,

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$

Let's first find $\overrightarrow{ON}$,

$$\overrightarrow{ON} = \frac{1}{2}\overrightarrow{OA} + \overrightarrow{OC} + \overrightarrow{CG}$$
$$\overrightarrow{ON} = \frac{1}{2} \times 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$
$$\overrightarrow{ON} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

Now let's evaluate $\overrightarrow{MN}$,

$$\overrightarrow{MN} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k} - 2\mathbf{i} - j$$
$$\overrightarrow{MN} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

Therefore, the final answer is,

$$\overrightarrow{OM} = 2\mathbf{i} + j \quad \overrightarrow{MN} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

(b) Find a vector equation for the line through M and N.

$$r = a + tb$$

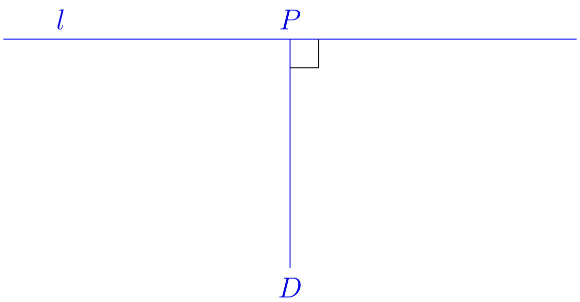
a is point on the line. Let's use $\overrightarrow{ON}$. b is the direction vector, in this case $\overrightarrow{MN}$. Therefore, the final answer is,

$$r = 2\mathbf{i} + j + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

(c) Find the position vector of P, the foot of the perpendicular from D to the line through M and N.

$$r = 2\mathbf{i} + j + t(-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

Let's sketch a diagram of our problem,



Note: l represents the line through M and N.

Let's find the general coordinates of P. We can use the equation for the line through M and N, since P lies on that line,

$$\overrightarrow{OP} = \begin{pmatrix} 2-t \\ 1+2t \\ 2t \end{pmatrix}$$

Now let's find the general coordinates of $\overrightarrow{DP}$,

$$\overrightarrow{DP} = \overrightarrow{OP} - \overrightarrow{OD}$$

$$\overrightarrow{DP} = \begin{pmatrix} 2-t \\ 1+2t \\ 2t \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\overrightarrow{DP} = \begin{pmatrix} 2-t \\ 1+2t \\ 2t-2 \end{pmatrix}$$

When two lines are perpendicular, their scalar product is equal to zero. This means that the scalar product of $\overrightarrow{DP}$ and the direction vector of the line through M and N,

$$\begin{pmatrix} 2-t \\ 1+2t \\ 2t-2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = 0$$
$$-2+t+2+4t+4t-4=0$$
$$-4+9t=0$$
$$9t=4$$
$$t=\frac{4}{9}$$

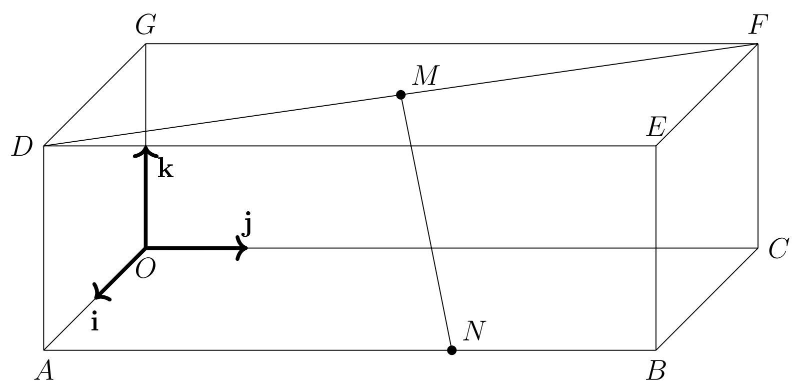
Now let's evaluate the coordinates of the position vector of P,

$$\overrightarrow{OP} = \begin{pmatrix} 2-\frac{4}{9} \\ 1+2\left(\frac{4}{9}\right) \\ 2\left(\frac{4}{9}\right) \end{pmatrix}$$
$$\overrightarrow{OP} = \begin{pmatrix} \frac{14}{9} \\ \frac{17}{9} \\ \frac{8}{9} \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OP} = \begin{pmatrix} \frac{14}{9} \\ \frac{17}{9} \\ \frac{8}{9} \end{pmatrix}$$

2.



In the diagram, $OABCDEFG$ is a cuboid in which $OA = 2$ units, $OC = 4$ units and $OG = 2$ units. Unit vectors **i**, **j** and **k** are parallel to OA, OC and OG respectively. The point M is the midpoint of DF. The point N on AB is such that $AN = 3NB$. (9709/31/M/J/22 number 9)

(a) Express the vectors $\overrightarrow{OM}$ and $\overrightarrow{MN}$ in terms of **i**, **j** and **k**.

We can write $\overrightarrow{OM}$ as,

$$\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC} + \overrightarrow{OG}$$
$$\overrightarrow{OM} = \frac{1}{2} \times 2\mathbf{i} + \frac{1}{2} \times 4\mathbf{j} + 2\mathbf{k}$$
$$\overrightarrow{OM} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

We can write $\overrightarrow{MN}$ as,

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$

Let's first find $\overrightarrow{ON}$,

$$\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN}$$
$$\overrightarrow{ON} = 2\mathbf{i} + \frac{3}{4}\overrightarrow{AB}$$

Note: Remember that $AN = 3NB$.

$$\overrightarrow{ON} = 2\mathbf{i} + \frac{3}{4} \times 4\mathbf{j}$$
$$\overrightarrow{ON} = 2\mathbf{i} + 3\mathbf{j}$$

Now let's go back to $\overrightarrow{MN}$,

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$
$$\overrightarrow{MN} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$
$$\overrightarrow{MN} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

Therefore, the final answer is,

$$\overrightarrow{OM} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \quad \overrightarrow{MN} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$$

(b) Find a vector equation for the line through M and N.

$$r = a + tb$$

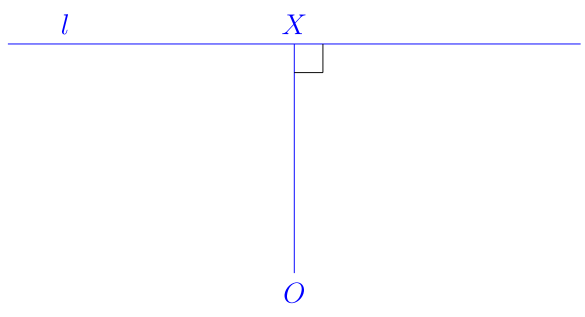
We will use the point $\overrightarrow{ON}$ and our direction vector $\overrightarrow{MN}$. Therefore, the final answer is,

$$r = 2\mathbf{i} + 3\mathbf{j} + t(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

(c) Show that the length of the perpendicular from O to the line through M and N is $\sqrt{\frac{53}{6}}$.

$$r = 2\mathbf{i} + 3\mathbf{j} + t(\mathbf{i} + \mathbf{j} - 2\mathbf{k})$$

Let's sketch a diagram of our problem,



Note: l represents the line through M and N.

Let's by finding the general coordinates of $\overrightarrow{OX}$ since it lies on the line,

$$\overrightarrow{OX} = \begin{pmatrix} 2+t \\ 3+t \\ -2t \end{pmatrix}$$

When two lines are perpendicular to each other their scalar product is equal to 0. In this case the line is perpendicular to $\overrightarrow{OX}$,

$$\begin{pmatrix} 2+t \\ 3+t \\ -2t \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$$

$$2 + t + 3 + t + 4t = 0$$

$$5 + 6t = 0$$

$$6t = 5$$

$$t = -\frac{5}{6}$$

Now let's find the exact coordinates of $\overrightarrow{OX}$,

$$\overrightarrow{OX} = \begin{pmatrix} 2 + \left(-\frac{5}{6}\right) \\ 3 + \left(-\frac{5}{6}\right) \\ -2\left(-\frac{5}{6}\right) \end{pmatrix}$$

$$\overrightarrow{OX} = \begin{pmatrix} \frac{7}{6} \\ \frac{13}{6} \\ \frac{5}{3} \end{pmatrix}$$

Now let's find the length of $\overrightarrow{OX}$,

$$|OX| = \sqrt{\left(\frac{7}{6}\right)^2 + \left(\frac{13}{6}\right)^2 + \left(\frac{5}{3}\right)^2}$$

$$|OX| = \sqrt{\frac{53}{6}}$$

Therefore, the final answer is,

$$|OX| = \sqrt{\frac{53}{6}}$$

3. The lines l and m have vector equations

$$r = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad r = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

respectively, where a and b are constants. (9709/32/M/J/22 number 9)

(a) Given that l and m intersect, show that $2b - a = 4$.

$$r = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad r = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

Let's write out the equations we get from the vector equations,

$$x = -1 + 2\lambda \quad x = 5 + a\mu$$
$$y = 3 - \lambda \quad y = 4 + b\mu$$
$$z = 4 - \lambda \quad z = 3 + \mu$$

Equate each set of equations,

$$-1 + 2\lambda = 5 + a\mu$$
$$3 - \lambda = 4 + b\mu$$
$$4 - \lambda = 3 + \mu$$

Let's make μ the subject of the formula in the third equation,

$$\mu = 1 - \lambda$$

Substitute into the second equation,

$$3 - \lambda = 4 + b(1 - \lambda)$$
$$3 - \lambda = 4 + b - b\lambda$$

Let's solve for λ in terms of b,

$$b\lambda - \lambda = 4 - 3 + b$$
$$\lambda(b - 1) = 1 + b$$
$$\lambda = \frac{1 + b}{b - 1}$$

Now evaluate μ in terms of b,

$$\mu = 1 - \frac{1 + b}{b - 1}$$
$$\mu = \frac{b - 1 - 1 - b}{b - 1}$$
$$\mu = \frac{-2}{b - 1}$$

Substitute μ and λ into the first equation,

$$-1 + 2\left(\frac{1 + b}{b - 1}\right) = 5 + a\left(\frac{-2}{b - 1}\right)$$

Get rid of the denominator,

$$-1(b - 1) + 2(1 + b) = 5(b - 1) - 2a$$
$$-b + 1 + 2 + 2b = 5b - 5 - 2a$$
$$b + 3 = 5b - 5 - 2a$$
$$5b - b - 2a = 5 + 3$$
$$4b - 2a = 8$$
$$2b - a = 4$$

Therefore, the final answer is,

$$2b - a = 4$$

(b) Given also that l and m are perpendicular, find the values of a and b.

$$r = -\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \lambda(2\mathbf{i} - \mathbf{j} - \mathbf{k}) \quad \text{and} \quad r = 5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + \mu(a\mathbf{i} + b\mathbf{j} + \mathbf{k})$$

If l and m are perpendicular then the scalar product of their direction vectors is equal to 0,

$$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ 1 \end{pmatrix} = 0$$

$$2a - b - 1 = 0$$

Now we have two equations in terms of a and b,

$$2b - a = 4 \quad 2a - b - 1 = 0$$

Let's solve them simultaneously,

$$a = 2b - 4$$

$$2a - b - 1 = 0$$

$$2(2b - 4) - b - 1 = 0$$

$$4b - 8 - b - 1 = 0$$

$$3b - 9 = 0$$

$$3b = 9$$

$$b = 3$$

Let's evaluate a,

$$a = 2(3) - 4$$

$$a = 2$$

Therefore, the final answer is,

$$a = 2 \quad b = 3$$

(c) When a and b have these values, find the position vector of the point of intersection of l and m.

$$x = -1 + 2\lambda$$

$$y = 3 - \lambda$$

$$z = 4 - \lambda$$

Our position vector has the general coordinates that we deduced in part (a). We also determined that,

$$\lambda = \frac{1 + b}{b - 1}$$

Evaluate λ,

$$\lambda = \frac{1+3}{3-1}$$
$$\lambda = 2$$

Substitute into the general coordinates,

$$x = -1 + 2(2)$$
$$y = 3 - 2$$
$$z = 4 - 2$$

This simplifies to give,

$$x = 3$$
$$y = 1$$
$$z = 2$$

Therefore, the final answer is,

$$3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

4. With respect to the origin O, the point A has position vector given by $\overrightarrow{OA} = \mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$. The line l has vector equation $r = 4\mathbf{i} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$. (9709/33/M/J/22 number 9)

   (a) Find in degrees the acute angle between the directions of OA and l.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

Let's find the scalar product of the directions of OA and l,

$$\begin{pmatrix}1\\5\\6\end{pmatrix} \cdot \begin{pmatrix}-1\\2\\3\end{pmatrix}$$
$$-1 + 10 + 18$$
$$27$$

Let's find the magnitude of each vector,

$$|OA| = \sqrt{1^2 + 5^2 + 6^2}$$
$$|OA| = \sqrt{62}$$
$$|l| = \sqrt{(-1)^2 + 2^2 + 3^2}$$
$$|l| = \sqrt{14}$$

Let's substitute into the formula,

$$\cos\theta = \frac{27}{\sqrt{62}\times\sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{27}{\sqrt{62}\times\sqrt{14}}\right)$$

$$\theta = 23.6^\circ$$

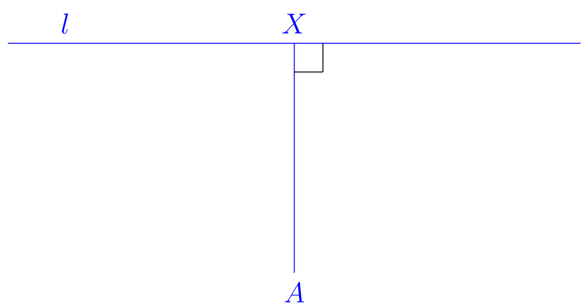
Therefore, the final answer is,

$$\theta = 23.6^\circ$$

(b) Find the position vector of the foot of the perpendicular from A to l.

$$r = 4\mathbf{i} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$$

Let's sketch a diagram of our problem,



Let's find the general coordinates of X since it lies on the line l,

$$\overrightarrow{OX} = \begin{pmatrix} 4-\lambda \\ 2\lambda \\ 1+3\lambda \end{pmatrix}$$

Now let's find the general coordinates of $\overrightarrow{AX}$,

$$\overrightarrow{AX} = \overrightarrow{OX} - \overrightarrow{OA}$$

$$\overrightarrow{AX} = \begin{pmatrix} 4-\lambda \\ 2\lambda \\ 1+3\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$$

$$\overrightarrow{AX} = \begin{pmatrix} 3-\lambda \\ -5+2\lambda \\ -5+3\lambda \end{pmatrix}$$

When two lines are perpendicular to each other, their scalar product is equal to 0. In this case $\overrightarrow{AX}$ and l are perpendicular,

$$\begin{pmatrix} 3-\lambda \\ -5+2\lambda \\ -5+3\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = 0$$

$$-3+\lambda-10+4\lambda-15+9\lambda = 0$$

$$-28+14\lambda = 0$$

$$14\lambda = 28$$

$$\lambda = 2$$

Now let's evaluate the coordinates of X,

$$\overrightarrow{OX} = \begin{pmatrix} 4-2 \\ 2(2) \\ 1+3(2) \end{pmatrix}$$

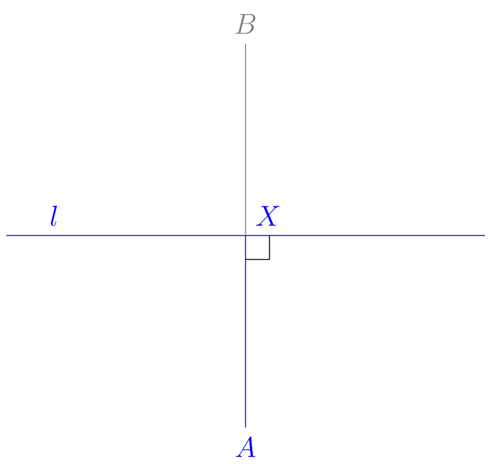
$$\overrightarrow{OX} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OX} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix}$$

(c) Hence find the position vector of the reflection of A in l.

If you reflect A in l you get B,



Since B is a reflection of A in the line l. This means that $\overrightarrow{AX}$ should be equal to $\overrightarrow{XB}$,

$$\overrightarrow{AX} = \overrightarrow{XB}$$
$$\overrightarrow{AX} = \overrightarrow{OB} - \overrightarrow{OX}$$

We want to find $\overrightarrow{OB}$, so make it the subject of the formula,

$$\overrightarrow{OB} = \overrightarrow{OX} + \overrightarrow{AX}$$

We already evaluated $\overrightarrow{OX}$ in part (b). We didn't finish evaluating $\overrightarrow{AX}$ in part (b), so let's evaluate it now that we know that $\lambda = 2$,

$$\overrightarrow{AX} = \begin{pmatrix} 3-\lambda \\ -5+2\lambda \\ -5+3\lambda \end{pmatrix}$$

$$\overrightarrow{AX} = \begin{pmatrix} 3-2 \\ -5+2(2) \\ -5+3(2) \end{pmatrix}$$

$$\overrightarrow{AX} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Now let's evaluate $\overrightarrow{OB}$,

$$\overrightarrow{OB} = \overrightarrow{OX} + \overrightarrow{AX}$$

$$\overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{OB} = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OB} = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}$$

5. With respect to the origin O, the position vectors of the points A, B and C are given by

$$\overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$$

The midpoint of AC is M and the point N lies on BC, between B and C, and is such that $BN = 2NC$. (9709/33/O/N/22 number 9)

(a) Find the position vectors of M and N.

Let's start with the position vector of M,

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OA}\right)$$

$$\overrightarrow{OM} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}\right)$$

$$\overrightarrow{OM} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \frac{1}{2}\left(\begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}\right)$$

$$\overrightarrow{OM} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$$

$$\overrightarrow{OM} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Now let's find the position vector of N,

$$\overrightarrow{BN} = 2\overrightarrow{NC}$$
$$\overrightarrow{ON} - \overrightarrow{OB} = 2\left(\overrightarrow{OC} - \overrightarrow{ON}\right)$$
$$\overrightarrow{ON} - \overrightarrow{OB} = 2\overrightarrow{OC} - 2\overrightarrow{ON}$$
$$3\overrightarrow{ON} = 2\overrightarrow{OC} + \overrightarrow{OB}$$
$$\overrightarrow{ON} = \frac{2}{3}\overrightarrow{OC} + \frac{1}{3}\overrightarrow{OB}$$

$$\overrightarrow{ON} = \frac{2}{3}\begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} \frac{8}{3} \\ -2 \\ -\frac{4}{3} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OM} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \overrightarrow{ON} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

(b) Find a vector equation for the line through M and N.

$$r = a + tb$$

We will use the point $\overrightarrow{OM}$. Let's find the direction vector $\overrightarrow{MN}$,

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{M}$$

$$\overrightarrow{MN} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$\overrightarrow{MN} = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

Therefore, the final answer is,

$$r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

(c) Find the position vector of the point Q where the line through M and N intersects the line through A and B.

$$r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$$

We already have the equation of the line through M and N. Let's find the equation of the line through A and B. Let's start by finding its direction vector $\overrightarrow{AB}$,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$$

This means the vector equation for the line through A and B is,

$$r = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$$

Now we have the two vector equations,

$$r = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \qquad r = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ -5 \\ -1 \end{pmatrix}$$

Let's solve them simultaneously,

$$2 + t = s$$

$$1 - 3t = 5 - 5s$$

$$-t = 2 - s$$

Substitute $s = 2 + t$ into the second equation,

$$1 - 3t = 5 - 5(2 + t)$$

$$1 - 3t = 5 - 10 - 5t$$

$$2t = 5 - 10 - 1$$

$$2t = -6$$

$$t = -3$$

The coordinates of the point of intersection Q are,

$$\overrightarrow{OQ} = \begin{pmatrix} 2+t \\ 1-3t \\ -t \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} 2-3 \\ 1-3(-3) \\ -(-3) \end{pmatrix}$$

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 10 \\ 3 \end{pmatrix}$$

6. With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \overrightarrow{OD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix}$$

(9709/32/F/M/23 number 10)

(a) Find the obtuse angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta = \frac{OA \cdot OB}{|OA||OB|}$$

Let's find their scalar product,

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

$$3 - 2 - 6$$

$$-5$$

Let's find their magnitudes,

$$|OA| = \sqrt{3^2 + (-1)^2 + 2^2}$$

$$|OA| = \sqrt{14}$$

$$|OB| = \sqrt{1^2 + 2^2 + (-3)^2}$$

$$|OB| = \sqrt{14}$$

Substitute into the formula,

$$\cos\theta = \frac{-5}{\sqrt{14} \times \sqrt{14}}$$

$$\theta = \cos^{-1}\left(\frac{-5}{\sqrt{14} \times \sqrt{14}}\right)$$

$$\theta = 110.9^\circ$$

Therefore, the final answer is,

$$\theta = 110.9^\circ$$

The line l passes through the points A and B.

(b) Find a vector equation for the line l.

$$r = a + tb$$

Let's start by finding the direction vector $\overrightarrow{AB}$,

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

Therefore, the final answer is,

$$r = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

(c) Find the position vector of the point of intersection of the line l and the line passing through C and D.

$$r = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix}$$

We already have the equation of line l. We need to find the equation of the line passing through C and D. Let's start by finding the direction vector $\overrightarrow{CD}$,

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$\overrightarrow{CD} = \begin{pmatrix} 5 \\ -6 \\ 11 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$\overrightarrow{CD} = \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

This means the equation of the line through C and D is,

$$r = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + s \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

Now we have the two vector equations,

$$r = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ -5 \end{pmatrix} \quad r = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + s \begin{pmatrix} 4 \\ -4 \\ 6 \end{pmatrix}$$

Let's solve the two equations simultaneously,

$$1 - 2t = 1 + 4s$$

$$2 + 3t = -2 - 4s$$

$$-3 - 5t = 5 + 6s$$

Let's make t the subject of the formula in the first equation,

$$2t = 1 - 1 - 4s$$

$$2t = -4s$$

$$t = -2s$$

Substitute $t = -2s$ into the second equation,

$$2 + 3(-2s) = -2 - 4s$$

$$2 - 6s = -2 - 4s$$

$$2s = 4$$

$$s = 2$$

The coordinates of the point of intersection are,

$$\begin{pmatrix} 1+4s \\ -2-4s \\ 5+6s \end{pmatrix}$$

Substitute s with 2,

$$\begin{pmatrix} 1+4(2) \\ -2-4(2) \\ 5+6(2) \end{pmatrix}$$

$$\begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$$

Therefore, the final answer is,

$$\begin{pmatrix} 9 \\ -10 \\ 17 \end{pmatrix}$$

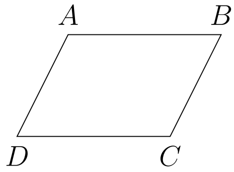
7. Relative to the origin, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \quad \overrightarrow{OB} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}, \quad \text{and} \quad \overrightarrow{OC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$$

The quadrilateral $ABCD$ is a parallelogram. (9709/31/M/J/23 number 6)

(a) Find the position vector of D.

Let's sketch a diagram of our problem,



You will notice that,

$$\overrightarrow{AB} = \overrightarrow{DC}$$
$$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$$
$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{OA} - \overrightarrow{OB}$$
$$\overrightarrow{OD} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$
$$\overrightarrow{OD} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$$

Therefore, the final answer is,

$$\overrightarrow{OD} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$$

(b) The angle between BA and BC is θ.
Find the exact value of $\cos\theta$.

$$\cos\theta = \frac{BA \cdot BC}{|BA||BC|}$$

Let's find $\overrightarrow{BA}$ and $\overrightarrow{BC}$,

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$$
$$\overrightarrow{BA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$
$$\overrightarrow{BA} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$
$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$\overrightarrow{BC} = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$
$$\overrightarrow{BC} = \begin{pmatrix} -1 \\ -5 \\ -6 \end{pmatrix}$$

Let's find their scalar product,

$$\begin{pmatrix}-2\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-5\\-6\end{pmatrix}$$

$$2+10-6$$

$$6$$

Let's find their magnitudes,

$$|BA| = \sqrt{(-2)^2+(-2)^2+1^2}$$

$$|BA| = 3$$

$$|BC| = \sqrt{(-1)^2+(-5)^2+(-6)^2}$$

$$|BC| = \sqrt{62}$$

Substitute into the formula,

$$\cos\theta = \frac{6}{3\sqrt{62}}$$

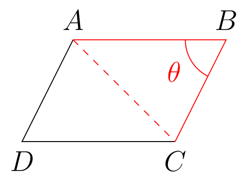
$$\cos\theta = \frac{2}{\sqrt{62}}$$

Therefore, the final answer is,

$$\cos\theta = \frac{2}{\sqrt{62}}$$

(c) Hence find the area of $ABCD$, giving your answers in the form $p\sqrt{q}$, where p and q are integers.

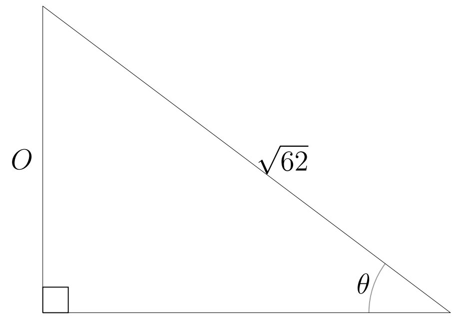
Let's sketch a diagram of our problem,



The area of the parallelogram is twice the area of the triangle. Let's find the area of the triangle,

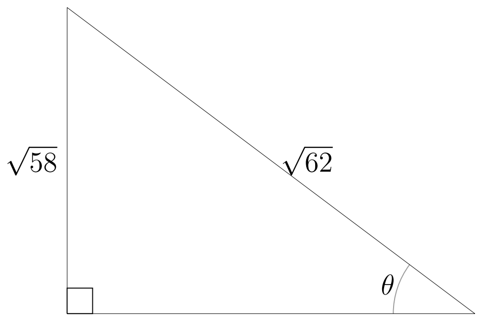
$$\frac{1}{2}\times 3\times\sqrt{62}\times\sin\theta$$

Let's find $\sin\theta$ first. To do that, let's draw a right angled triangle and use $SOHCAHTOA$ and the fact that $\cos\theta = \frac{2}{\sqrt{62}}$,



Let's find the length of the opposite side,

$$O = \sqrt{\sqrt{62}^2 - 2^2}$$
$$O = \sqrt{58}$$



Let's read off the exact value of $\sin\theta$ from the diagram,

$$\sin\theta = \frac{\sqrt{58}}{\sqrt{62}}$$

Now let's go back to the triangle inside the parallelogram,

$$\frac{1}{2} \times 3 \times \sqrt{62} \times \sin\theta$$
$$\frac{1}{2} \times 3 \times \sqrt{62} \times \frac{\sqrt{58}}{\sqrt{62}}$$
$$\frac{3\sqrt{58}}{2}$$

Remember that 2 of these triangles make up the parallelogram,

$$2 \times \frac{3\sqrt{58}}{2}$$
$$3\sqrt{58}$$

Therefore, the final answer is,

$$3\sqrt{58}$$