Pure Maths 3

3.8 Differential Equations - Easy

Subject:	**Mathematics**
Syllabus Code:	**9709**
Level:	**A2 Level**
Component:	**Pure Mathematics 3**
Topic:	**3.8 Differential Equations**
Difficulty:	**Easy**

Questions

1. The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{dy}{dx} = \left(1 - 2x^2\right) y$$

for $x > 0$. It is given that $y = 1$ when $x = 1$. (9709/31/O/N/20 number 8)

Solve the differential equation, obtaining an expression for y in terms of x.

2. The variables x and y satisfy the differential equation

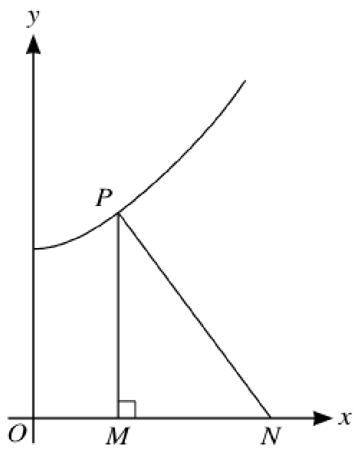
$$(1 - \cos x)\frac{dy}{dx} = y \sin x$$

It is given that $y = 4$ when $x = \pi$. (9709/32/F/M/21 number 4)

(a) Solve the differential equation, obtaining an expression for y in terms of x.

(b) Sketch the graph of y against x for $0 < x < 2\pi$.

3.



For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perpendicular from P to the x-axis.

The curve is such that for all values of x in the interval $0 \leq x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$. (9709/33/M/J/21 number 7)

(a) i. Show that $\frac{MN}{y} = \frac{dy}{dx}$.

ii. Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\frac{dy}{dx} = \tan x$.

(b) Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing y in terms of x.

4. A large plantation of area 20 km^2 is becoming infected with a plant disease. At time t years the area infected is x km^2 and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected.

When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$. (9709/33/O/N/21 number 10)

(a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}$$

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$.

(c) Use an iterative formula based on the equation in part (a), with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

(d) Calculate the value of t at which the entire plantation becomes infected.

5. The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1 + x^2}$$

and $y = 2$ when $x = 0$. (9709/31/M/J/22 number 4)

Solve the differential equation, obtaining a simplified expression for y in terms of x.

6. At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}} \cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$. (9709/33/M/J/22 number 8)

(a) Solve the differential equation, obtaining a relation between N, k and t.

(b) Given also that $N = 625$ when $t = 50$, find the value of k.

(c) Obtain an expression for n in terms of t, and find the greatest value of N predicted by this model.

7. In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t}$$

where k is a positive constant. It is given that $x = 20$ at the start of the reaction. (9709/31/O/N/22 number 8)

(a) Solve the differential equation, obtaining a relation between x, t and k.

(b) Given that $x = 40$ when $t = 10$, find the value of k and find the value approached by x as t becomes large.

8. A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

The differential equation satisfied by V and t is of the form $\frac{dV}{dt} = a - bV$. (9709/33/O/N/22 number 10)

(a) Write down the values of the constants a and b.

(b) Solve the differential equation and find the value of t when $V = 1000$.

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large.

Answers

1. The coordinates (x, y) of a general point of a curve satisfy the differential equation

$$x\frac{dy}{dx} = \left(1 - 2x^2\right) y$$

for $x > 0$. It is given that $y = 1$ when $x = 1$. (9709/31/O/N/20 number 8)

Solve the differential equation, obtaining an expression for y in terms of x.

$$x\frac{dy}{dx} = \left(1 - 2x^2\right) y$$

Let's separate the variables,

$$\frac{1}{y}dy = \frac{1 - 2x^2}{x}dx$$

$$\frac{1}{y}dy = \frac{1}{x} - \frac{2x^2}{x}\ dx$$

$$\frac{1}{y}dy = \frac{1}{x} - 2x\ dx$$

Integrate both sides,

$$\int \frac{1}{y}dy = \int \frac{1}{x} - 2x\ dx$$

$$\ln y + c = \ln x - x^2$$

Let's evaluate y using $y = 1$ when $x = 1$,

$$\ln 1 + c = \ln 1 - (1)^2$$

$$c = 1$$

Substitute c,

$$\ln y - 1 = \ln x - x^2$$

Make y the subject of the formula,

$$\ln y = \ln x + 1 - x^2$$

$$y = e^{\ln x + 1 - x^2}$$

$$y = e^{\ln x} e^{1 - x^2}$$

$$y = xe^{1 - x^2}$$

Therefore, the final answer is,

$$y = xe^{1-x^2}$$

2. The variables x and y satisfy the differential equation

$$(1-\cos x)\frac{dy}{dx} = y\sin x$$

It is given that $y = 4$ when $x = \pi$. (9709/32/F/M/21 number 4)

(a) Solve the differential equation, obtaining an expression for y in terms of x.

$$(1-\cos x)\frac{dy}{dx} = y\sin x$$

Separate the variables,

$$\frac{1}{y}dy = \frac{\sin x}{1-\cos x}dx$$

Integrate both sides,

$$\int \frac{1}{y}dy = \int \frac{\sin x}{1-\cos x}dx$$
$$\ln y + c = \ln(1-\cos x)$$

Use $y = 4$ when $x = \pi$ to evaluate c,

$$\ln 4 + c = \ln(1-\cos \pi)$$
$$\ln 4 + c = \ln(1-(-1))$$
$$\ln 4 + c = \ln 2$$
$$c = \ln 2 - \ln 4$$
$$c = \ln\frac{2}{4}$$
$$c = \ln\frac{1}{2}$$

Substitute c,

$$\ln y + \ln\frac{1}{2} = \ln(1-\cos x)$$

Let's make y the subject of the formula,

$$\ln y = \ln(1-\cos x) - \ln\frac{1}{2}$$

Combine the two logarithms on the right hand side,

$$\ln y = \ln\left(\frac{1-\cos x}{\frac{1}{2}}\right)$$
$$\ln y = \ln\left(2(1-\cos x)\right)$$
$$\ln y = \ln(2-2\cos x)$$

Apply the exponential function to both sides,

$$y = 2 - 2\cos x$$

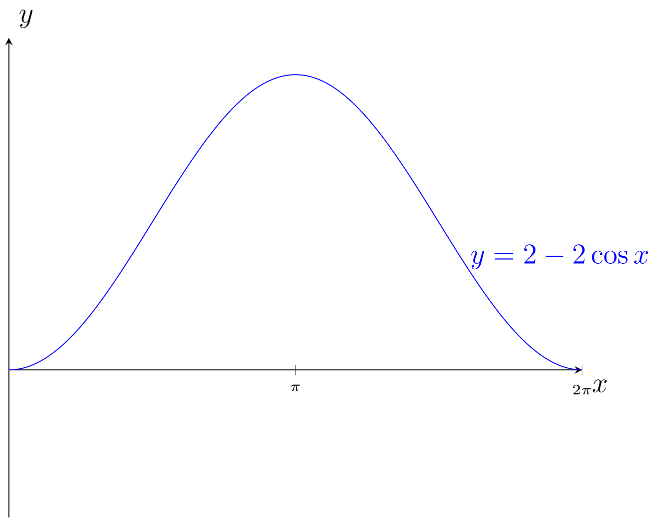
Therefore, the final answer is,

$$y = 2 - 2\cos x$$

(b) Sketch the graph of y against x for $0 < x < 2\pi$.

$$y = 2 - 2\cos x$$

Sketch the graph of $y = 2 - 2\cos x$**,**



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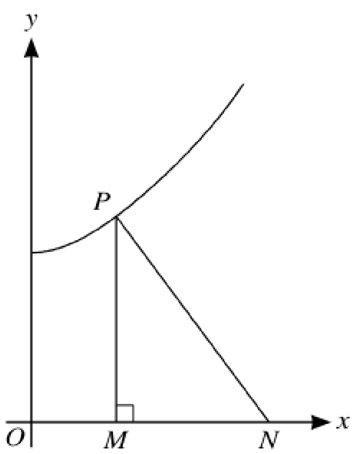

For the curve shown in the diagram, the normal to the curve at the point P with coordinates (x, y) meets the x-axis at N. The point M is the foot of the perpendicular from P to the x-axis.

The curve is such that for all values of x in the interval $0 \leq x < \frac{1}{2}\pi$, the area of triangle PMN is equal to $\tan x$. (9709/33/M/J/21 number 7)

(a) i. Show that $\frac{MN}{y} = \frac{dy}{dx}$.

Let's find the gradient of the normal through P and N,

$$m_N = \frac{y_1 - y_2}{x_1 - x_2}$$

$$m_N = \frac{0 - y}{MN}$$

From the diagram, we can tell that the gradient is supposed to be negative because of how the normal is sloped. This means that the gradient of our normal is,

$$m_N = -\frac{y}{MN}$$

$\frac{dy}{dx}$ denotes the gradient of the tangent. The gradient of the tangent is the negative reciprocal of the gradient of the normal,

$$m_T = \frac{MN}{y}$$

$$\frac{dy}{dx} = \frac{MN}{y}$$

Therefore, the final answer is,

$$\frac{MN}{y} = \frac{dy}{dx}$$

ii. Hence show that x and y satisfy the differential equation $\frac{1}{2}y^2\frac{dy}{dx} = \tan x$.

$$\frac{MN}{y} = \frac{dy}{dx}$$

We can write MN as,

$$MN = y\frac{dy}{dx}$$

Now using the triangle, let's create an equation based on its area,

$$\frac{1}{2} \times y \times MN = \tan x$$

Substitute MN,

$$\frac{1}{2} \times y \times y\frac{dy}{dx} = \tan x$$

$$\frac{1}{2}y^2\frac{dy}{dx} = \tan x$$

Therefore, the final answer is,

$$\frac{1}{2}y^2\frac{dy}{dx} = \tan x$$

(b) Given that $y = 1$ when $x = 0$, solve this differential equation to find the equation of the curve, expressing y in terms of x.

$$\frac{1}{2}y^2\frac{dy}{dx} = \tan x$$

Let's separate the variables,

$$\frac{1}{2}y^2dy = \tan x dx$$

Integrate both sides,

$$\int \frac{1}{2}y^2dy = \int \tan x dx$$

$$\frac{1}{6}y^3 + c = -\int \frac{-\sin x}{\cos x}dx$$

$$\frac{1}{6}y^3 + c = -\ln\cos x$$

Use $y = 1$ when $x = 0$ to evaluate c,

$$\frac{1}{6}(1)^3 + c = -\ln\cos 0$$

$$\frac{1}{6} + c = -\ln(1)$$

$$\frac{1}{6} + c = 0$$

$$c = -\frac{1}{6}$$

Substitute c,

$$\frac{1}{6}y^3 - \frac{1}{6} = -\ln\cos x$$

Make y the subject of the formula,

$$\frac{1}{6}y^3 = \frac{1}{6} - \ln\cos x$$

$$y^3 = 1 - 6\cos x$$

$$y = (1 - 6\cos x)^{\frac{1}{3}}$$

Therefore, the final answer is,

$$y = (1 - 6\cos x)^{\frac{1}{3}}$$

4. A large plantation of area 20 km^2 is becoming infected with a plant disease. At time t years the area infected is x km^2 and the rate of increase of x is proportional to the ratio of the area infected to the area not yet infected.

   When $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$. (9709/33/O/N/21 number 10)

   (a) Show that x and t satisfy the differential equation

$$\frac{dx}{dt} = \frac{19x}{20 - x}$$

We are told that "rate of increase of x is proportional to the ratio of the area infected to the area not yet infected",

$$\frac{dx}{dt} \propto \frac{x}{20 - x}$$
$$\frac{dx}{dt} = k\frac{x}{20 - x}$$

We are told that when $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$,

$$1 = k\frac{1}{20 - 1}$$
$$1 = \frac{1}{19}k$$
$$k = 19$$

This means that,

$$\frac{dx}{dt} = 19\frac{x}{20 - x}$$
$$\frac{dx}{dt} = \frac{19x}{20 - x}$$

Therefore, the final answer is,

$$\frac{dx}{dt} = \frac{19x}{20 - x}$$

(b) Solve the differential equation and show that when $t = 1$ the value of x satisfies the equation $x = e^{0.9+0.05x}$.

$$\frac{dx}{dt} = \frac{19x}{20 - x}$$

Separate the variables,

$$\frac{20 - x}{19x}dx = 1\ dt$$
$$\frac{20}{19x} - \frac{x}{19x}\ dx = 1\ dt$$
$$\frac{20}{19x} - \frac{1}{19}\ dx = 1\ dt$$

Integrate both sides,

$$\int \frac{20}{19x} - \frac{1}{19}\ dx = \int 1\ dt$$
$$\frac{20}{19}\ln x - \frac{1}{19}x + c = t$$

Use $t = 0$ when $x = 1$ to evaluate c,

$$\frac{20}{19}\ln 1 - \frac{1}{19}(1) + c = 0$$
$$-\frac{1}{19} + c = 0$$
$$c = \frac{1}{19}$$

Substitute c,

$$\frac{20}{19}\ln x - \frac{1}{19}x + \frac{1}{19} = t$$

Simplify,

$$20\ln x - x + 1 = 19t$$
$$20\ln x = 19t - 1 + x$$

Substitute t with 1,

$$20\ln x = 19(1) - 1 + x$$
$$20\ln x = 18 + x$$

Divide both sides by 20,

$$\ln x = 0.9 + 0.05x$$

Apply the exponential function to both sides,

$$x = e^{0.9+0.05x}$$

Therefore, the final answer is,

$$x = e^{0.9+0.05x}$$

(c) Use an iterative formula based on the equation in part (a), with an initial value of 2, to determine x correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

$$x_{n+1} = e^{0.9+0.05x_n}$$

We are told that our initial value, x_0 is 2,

$$x_1 = e^{0.9+0.05(2)} = 2.7183$$
$$x_2 = e^{0.9+0.05(Ans)} = 2.8177$$
$$x_3 = 2.8317 \approx 2.83$$
$$x_4 = 2.8337 \approx 2.83$$

Therefore, the final answer is,

$$x = 2.83$$

(d) Calculate the value of t at which the entire plantation becomes infected.

$$20\ln x = 19t - 1 + x$$

We are told that the entire plantation is 20 km ^{2}. This means that it becomes completely infected when $x = 20$,

$$20\ln 20 = 19t - 1 + 20$$
$$20\ln 20 = 19t + 19$$
$$19t = 20\ln 20 - 19$$
$$t = \frac{20\ln 20 - 19}{19}$$
$$t = 2.15$$

Therefore, the final answer is,

$$t = 2.15$$

5. The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{xy}{1+x^2}$$

and $y = 2$ when $x = 0$. (9709/31/M/J/22 number 4)

Solve the differential equation, obtaining a simplified expression for y in terms of x.

$$\frac{dy}{dx} = \frac{xy}{1+x^2}$$

Separate the variables,

$$\frac{1}{y}dy = \frac{x}{1+x^2}dx$$

Integrate both sides,

$$\int \frac{1}{y}dy = \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$\ln y + c = \frac{1}{2}\ln\left(1+x^2\right)$$

Use $y = 2$ when $x = 0$ to evaluate c,

$$\ln 2 + c = \frac{1}{2}\ln\left(1+(0)^2\right)$$

$$\ln 2 + c = \frac{1}{2}\ln(1)$$

$$\ln 2 + c = 0$$

$$c = -\ln 2$$

Substitute c,

$$\ln y - \ln 2 = \frac{1}{2}\ln\left(1+x^2\right)$$

Make y the subject of the formula,

$$\ln y = \frac{1}{2}\ln\left(1+x^2\right) + \ln 2$$

Combine the two logarithms on the right,

$$\ln y = \ln\left(\sqrt{1+x^2}\right) + \ln 2$$

$$\ln y = \ln\left(2\sqrt{1+x^2}\right)$$

Apply the exponential function to both sides,

$$y = 2\sqrt{1+x^2}$$

Therefore, the final answer is,

$$y = 2\sqrt{1+x^2}$$

6. At time t days after the start of observations, the number of insects in a population is N. The variation in the number of insects is modelled by a differential equation of the form $\frac{dN}{dt} = kN^{\frac{3}{2}}\cos 0.02t$, where k is a constant and N is a continuous variable. It is given that when $t = 0$, $N = 100$. (9709/33/M/J/22 number 8)

   (a) Solve the differential equation, obtaining a relation between N, k and t.

$$\frac{dN}{dt} = kN^{\frac{3}{2}}\cos 0.02t$$

Separate the variables,

$$\frac{1}{N^{\frac{3}{2}}}dN = k\cos 0.02tdt$$
$$N^{-\frac{3}{2}}dN = k\cos 0.02tdt$$

Integrate both sides,

$$\int N^{-\frac{3}{2}}dN = \int k\cos 0.02tdt$$
$$-2N^{-\frac{1}{2}} + c = \frac{k}{0.02}\sin 0.02t$$
$$-\frac{2}{\sqrt{N}} + c = 50k\sin 0.02t$$

Use $t = 0$ and $N = 100$ to evaluate c,

$$-\frac{2}{\sqrt{100}} + c = 50k\sin 0.02(0)$$
$$-\frac{2}{10} + c = 0$$
$$-\frac{1}{5} + c = 0$$
$$c = \frac{1}{5}$$

Substitute c,

$$-\frac{2}{\sqrt{N}} + \frac{1}{5} = 50k\sin 0.02t$$

Therefore, the final answer is,

$$-\frac{2}{\sqrt{N}} + \frac{1}{5} = 50k\sin 0.02t$$

   (b) Given also that $N = 625$ when $t = 50$, find the value of k.

$$-\frac{2}{\sqrt{N}} + \frac{1}{5} = 50k\sin 0.02t$$

Substitute the given values,

$$-\frac{2}{\sqrt{625}}+\frac{1}{5}=50k\sin 0.02(50)$$

$$-\frac{2}{25}+\frac{1}{5}=50k\sin 1$$

Make k the subject of the formula,

$$k=\frac{-\frac{2}{25}+\frac{1}{5}}{50\sin 1}$$

$$k=0.00285$$

Therefore, the final answer is,

$$k=0.00285$$

(c) Obtain an expression for n in terms of t, and find the greatest value of N predicted by this model.

$$-\frac{2}{\sqrt{N}}+\frac{1}{5}=50k\sin 0.02t$$

$$-\frac{2}{\sqrt{N}}+\frac{1}{5}=50(0.002852)\sin 0.02t$$

$$-\frac{2}{\sqrt{N}}+\frac{1}{5}=0.1426\sin 0.02t$$

Get rid of the denominators,

$$-10+\sqrt{N}=0.713\sin 0.02t\sqrt{N}$$

Put all the terms containing $\sqrt{N}$ on one side,

$$0.713\sin 0.02t\sqrt{N}-\sqrt{N}=-10$$

Factor out $\sqrt{N}$,

$$\sqrt{N}(0.713\sin 0.02t-1)=-10$$

Make $\sqrt{N}$ the subject of the formula,

$$\sqrt{N}=-\frac{10}{0.713\sin 0.02t-1}$$

Square both sides,

$$N=\left(-\frac{10}{0.713\sin 0.02t-1}\right)^2$$

The greatest value of $\sin 0.02t$ is 1,

$$N=\left(-\frac{10}{0.713(1)-1}\right)^2$$

This simplifies to give,

$$N = 1214$$

Therefore, the final answer is,

$$N = 1214$$

7. In a certain chemical reaction the amount, x grams, of a substance is increasing. The differential equation satisfied by x and t, the time in seconds since the reaction began, is

$$\frac{dx}{dt} = kxe^{-0.1t}$$

where k is a positive constant. It is given that $x = 20$ at the start of the reaction. (9709/31/O/N/22 number 8)

(a) Solve the differential equation, obtaining a relation between x, t and k.

$$\frac{dx}{dt} = kxe^{-0.1t}$$

Separate the variables,

$$\frac{1}{x}dx = ke^{-0.1t}dt$$

Integrate both sides,

$$\int \frac{1}{x}dx = \int ke^{-0.1t}dt$$

$$\ln x + c = -10ke^{-0.1t}$$

At the start of the reaction $x = 20$ and the time, $t = 0$,

$$\ln 20 + c = -10ke^{-0.1(0)}$$

$$\ln 20 + c = -10k$$

$$c = -10k - \ln 20$$

Substitute c,

$$\ln x - 10k - \ln 20 = -10ke^{-0.1t}$$

Therefore, the final answer is,

$$\ln x - 10k - \ln 20 = -10ke^{-0.1t}$$

(b) Given that $x = 40$ when $t = 10$, find the value of k and find the value approached by x as t becomes large.

$$\ln x - 10k - \ln 20 = -10ke^{-0.1t}$$

Substitute the given values,

$$\ln 40 - 10k - \ln 20 = -10ke^{-0.1(10)}$$

$$\ln \frac{40}{20} - 10k = -10ke^{-1}$$

$$\ln 2 - 10k = -10ke^{-1}$$

Put all the terms containing x on one side,

$$10ke^{-1} - 10k = -\ln 2$$

$$k\left(10e^{-1} - 10\right) = -\ln 2$$

$$k = -\frac{\ln 2}{10e^{-1} - 10}$$

As t becomes large,

$$\ln x - 10k - \ln 20 = -10ke^{-0.1t}$$

$e^{-0.1t}$ **or** $\frac{1}{e^{0.1t}}$ **approaches** 0**,**

$$\ln x - 10k - \ln 20 = -10k(0)$$

$$\ln x - 10k - \ln 20 = 0$$

Make $\ln x$ the subject of the formula,

$$\ln x = \ln 20 + 10k$$

$$x = e^{\ln 20 + 10k}$$

$$x = 20e^{10k}$$

$$x = 20\exp(10k)$$

Substitute k,

$$x = 20\exp\left(10 \times -\frac{\ln 2}{10e^{-1} - 10}\right)$$

$$x = 59.9$$

Therefore, the final answer is,

x approaches 59.9.

8. A gardener is filling an ornamental pool with water, using a hose that delivers 30 litres of water per minute. Initially the pool is empty. At time t minutes after filling begins the volume of water in the pool is V litres. The pool has a small leak and loses water at a rate of $0.01V$ litres per minute.

   The differential equation satisfied by V and t is of the form $\frac{dV}{dt} = a - bV$. (9709/33/O/N/22 number 10)

   (a) Write down the values of the constants a and b.

   Using the given information we can say that,

$$\frac{dV}{dt} = 30 - 0.01V$$

Therefore, the final answer is,

$$a = 30 \quad b = 0.01$$

(b) Solve the differential equation and find the value of t when $V = 1000$.

$$\frac{dV}{dt} = 30 - 0.01V$$

Separate the variables,

$$\frac{1}{30 - 0.01V}\ dV = 1\ dt$$

Integrate both sides,

$$-\frac{1}{0.01}\ln(30 - 0.01V) + c = t$$
$$-100\ln(30 - 0.01V) + c = t$$

We know that initially (at $t = 0$), the pool is empty ($V = 0$). We can use that knowledge to evaluate c,

$$-100\ln(30 - 0.01(0)) + c = 0$$

$$-100\ln(30) + c = 0$$
$$c = 100\ln(30)$$

Substitute c,

$$-100\ln(30 - 0.01V) + 100\ln(30) = t$$

Now let's evaluate t when $V = 1000$,

$$t = -100\ln(30 - 0.01(1000)) + 100\ln(30)$$

$$t = -100\ln(20) + 100\ln(30)$$
$$t = 40.5$$

Therefore, the final answer is,

$$t = 40.5$$

(c) Obtain an expression for V in terms of t and hence state what happens to V as t becomes large.

$$-100\ln(30 - 0.01V) + 100\ln(30) = t$$

Let's make V the subject of the formula,

$$100\ln(30 - 0.01V) = 100\ln(30) - t$$
$$\ln\left((30 - 0.01V)^{100}\right) = \ln 30^{100} - t$$
$$(30 - 0.01V)^{100} = e^{\ln 30^{100} - t}$$
$$(30 - 0.01V)^{100} = e^{\ln 30^{100}}e^{-t}$$
$$(30 - 0.01V)^{100} = 30^{100}e^{-t}$$

Take the 100-th root of both sides to simplify,

$$30 - 0.01V = 30e^{-100t}$$

$$0.01V = 30 - 30e^{-100t}$$

$$V = \frac{30 - 30e^{-100t}}{0.01}$$

$$V = 3000 - 3000e^{-100t}$$

$$V = 3000\left(1 - e^{-t}\right)$$

As t becomes large, e^{-t} or $\frac{1}{e^t}$ approaches 0,

$$V = 3000\,(1 - 0)$$

$$V = 3000$$

Therefore, the final answer is,

As t becomes large, V approaches 3000.